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THE

ELEMENTS

OF THE

CONIC SECTIONS

WITH

The Sections

OF THE

CONOIDS.

BY

JAMES DEVEREUX HUSTLER, B.D., F.R.S.,

LATE FELLOW AND TUTOR OF TRINITY COLLEGE.

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It has been the Editor's object in these Elements to provide short and easy proofs of all the Propositions in Conics which are required for Newton's Principia. The method of deriving the chief properties of the Ellipse and Hyperbola directly from the Cone was communicated to the Royal Society in 1843 by Sir Frederick Pollock, F.R.S., and is here introduced with his permission.

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THE

ELEMENTS

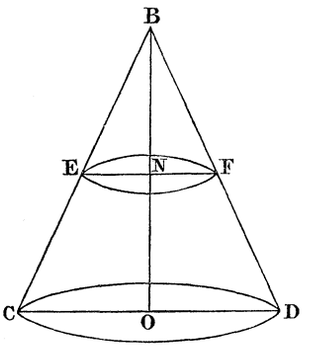
OF THE

CONIC SECTIONS.

On the Parabola.

DEFINITIONS.

1. THE solid generated by the revolution of a right-angled triangle *BOC* about either of its sides *BO* as an axis, is a *Right Cone.*

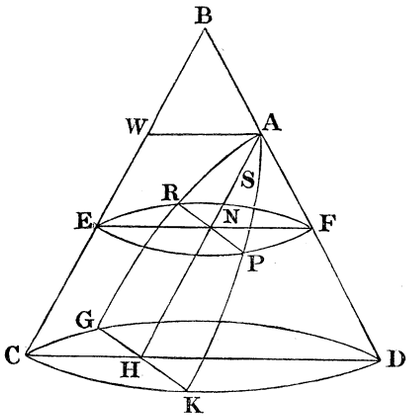


SCHOLIUM. The other side *CO* describes a circle *COD*, whose center *O* is in the axis *BO*, and to whose plane the axis is perpendicular. So also any other

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line NE parallel to OC in the revolving triangle describes a circle having its center N in the axis, and its plane perpendicular to the axis. Hence conversely, every section of the Cone perpendicular to the axis is a circle, having its center in the axis.

2. If a right cone be cut by a plane AGK which is parallel to a plane touching the cone along the slant side BC, the section AGK is called a *Parabola*.

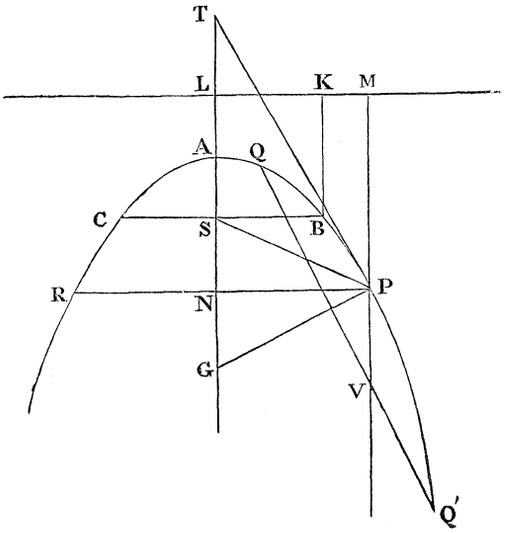


3. If BCD be that position of the revolving triangle which is perpendicular to the cutting plane AGK, their common section AH, which is parallel to BC (Eucl. 16. XI.) is called the *Axis* of the Parabola, and the point A, the *Vertex*.

4. If AW be drawn parallel to CD, and a point S be taken in the axis AH, so that $4AS$ may be a third proportional to BW and AW, the point S is called the *Focus*.

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5. If the axis be produced to L, so that AL may be equal to AS, the perpendicular LM to the axis at the point L is called the *Directrix*.



6. Any line PNR perpendicular to the axis and terminated both ways by the curve, is called an *Ordinate* to the axis; the part AN of the axis between the vertex and ordinate, the *Abscissa;* and the ordinate BC through the focus, the *Latus Rectum*.

7. Any line MPV parallel to the axis, is called a *Diameter;* a line QVQ' parallel to the tangent at any point P, is called an *Ordinate* to the point P or diameter PV; and the part PV of the diameter, the *Abscissa*. Also that ordinate, which passes through the focus, is called the *Parameter*.

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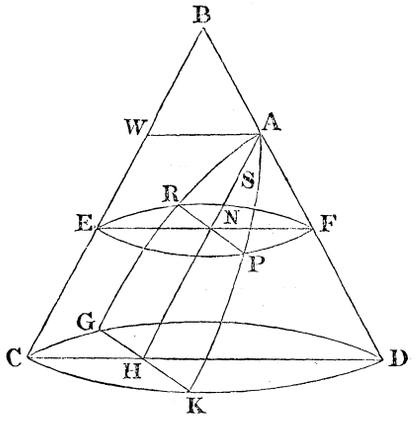
8. A straight line PG perpendicular to the tangent at any point P and terminated by the axis, is called a *Normal;* and the part NG of the axis, the *Subnormal.*

9. The part of any diameter between one of its own ordinates and the intersection of a tangent at the extremity of the ordinate, is called the *Subtangent.* Thus NT is the subtangent to the axis.

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Prop. I.

The rectangle contained by 4AS *and the abscissa* AN *is equal to the square of the semi-ordinate* NP.



Let $ENPF$ be any circular section, which being perpendicular to the axis is perpendicular to the plane BCD passing through the axis (Eucl. 18. xi.): also the cutting plane GAK is perpendicular to BCD. Hence the common section PNR is perpendicular to AH (Eucl. 19. xi.), and likewise to EF the diameter of the circle, which therefore bisects PR in F. (Eucl. 3. iii.)

But $AN : NF :: BW : AW :: AW(EN) : 4AS$ (Def. 4.)

$$\therefore 4AS \times AN = EN \times NF = NP^2,$$

by the property of the circle.

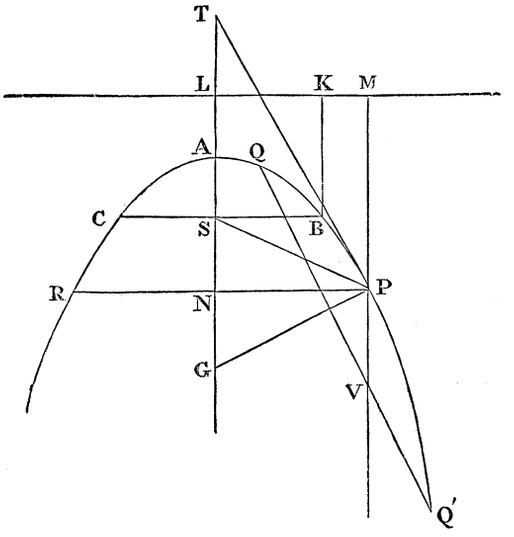
Cor. 1. The axis bisects all its ordinates.

Cor. 2. $AN \propto NP^2$.

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Prop. II.

The distance SP *of any point* P *in the parabola from the focus is equal to the perpendicular distance* PM *of the same point from the directrix.*



For $4AS \times AN + SN^2 = NL^2$ (Eucl. 8. II.)

Or $NP^2 + SN^2 (SP^2) = PM^2$. Therefore $SP = PM$.

Cor 1. The latus rectum BSC is equal to $4AS$. For draw BK perpendicular to the directrix. Then

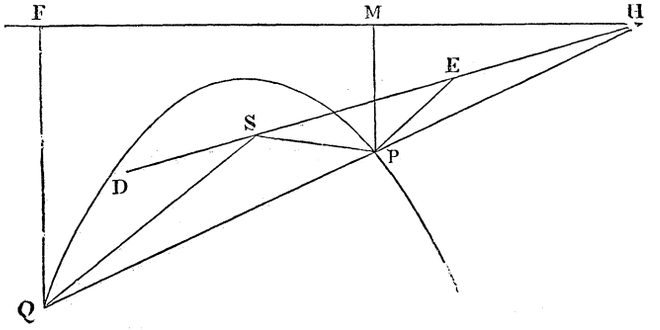
$SB = BK = SL = AL + AS = 2AS$ (Def. 5.)

Cor. 2. The rectangle by the latus rectum BC and the abscissa AN is equal to the square of the semi-ordinate NP.

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Prop. III.

If a straight line QP *cut the parabola in* P, Q, *and the directrix in* H, *then* HSD *drawn through the focus makes equal angles with the focal distances* SP, SQ.



Draw *PM*, *QF*, perpendicular to the directrix, and *PE* parallel to *SQ*; then the triangles *HPE*, *HQS*, and the triangles *HMP*, *HQF*, are similar.

Hence $PE : QS :: HP : HQ :: PM : QF$;

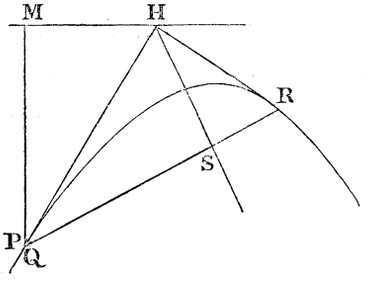
But $QS = QF$, and $\therefore PE = PM = SP$.

Hence $\angle PSE = \angle PES = \angle QSD$. (Eucl. 29. 1.)

Cor 1. If any straight line *HP*, not parallel to the axis, cuts a parabola in one point *P*, it will cut it again. For with center *P* and radius *PS* or *PM*, which is necessarily less than *PH*, suppose a circle described cutting *HS* in *E*. In *HP*, produced if necessary, take *HQ* a fourth proportional to *HE*, *HS*, *HP*; and draw *PM*, *QF*, perpendicular to the directrix. Then *Q* is a point in the curve, for $SQ = QF$, by similar triangles, as in the proposition.

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Cor. 2. If PQ move parallel to itself until the



points P and Q coincide, and HQ touches the parabola, the $\angle PSQ$ vanishes, and the angles made by SP, SQ with SH are right angles. Conversely, if SH be drawn perpendicular to any focal distance SP, cutting the directrix in H, then HP being joined touches the curve at P.

Cor. 3. Let PS meet the curve again in R; then HR being joined touches at R. The tangents therefore at the extremities of any parameter meet in the directrix. In the case of the latus rectum, the tangents meet in the intersection of the axis with the directrix, and are at right angles to each other.

Prop. IV.

A tangent PH *bisects the angle* SPM*.

For $\angle HSP$ is a right angle (Prop. 3. Cor. 2.); and HP is common to the two right-angled triangles

* The reference is always to the Figure immediately preceding.

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HSP, *HMP*; also $SP = PM$; therefore the triangles are equal, and $\angle SPH = \angle MPH$.

Cor. A tangent at the vertex is perpendicular to the axis, and parallel to all the ordinates of the axis.

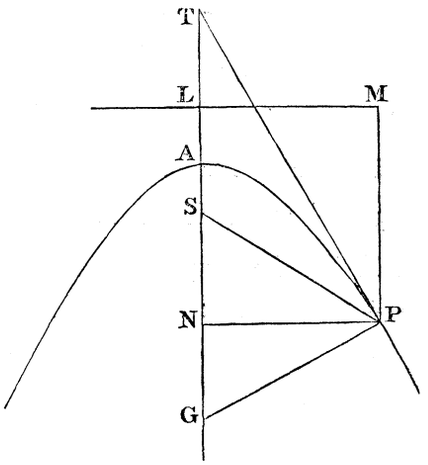
Prop. V.

The focal distance SP *is equal to*

I. *The abscissa of the axis* AN + AS.

II. *The distance* ST *of the tangent's intersection with the axis.*

III. *The distance* SG *of the normal's intersection with the axis.*



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For I. $SP = PM = NL = AN + AL = AN + AS$.

II. $\angle SPT = \angle MPT =$ altern. $\angle STP$; $\therefore SP = ST$.

III. The right angle $GPT = \angle PGT + \angle GTP$ $= \angle PGS + \angle SPT$; take away $\angle SPT$, and $\angle SPG = \angle SGP$; $\therefore SP = SG$.

PROP. VI.

The subnormal NG = 2 AS.

For $SG = SP = AN + AS$ (Prop. 5.) $= SN + 2AS$;

Take away SN, and $NG = 2AS$.

PROP. VII.

The subtangent NT *of the axis is double of the abscissa* AN.

For $ST = SP = AN + AS$ (Prop. 5.);

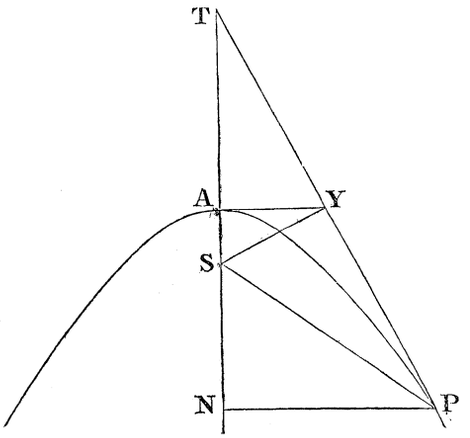
Take away AS, and $AT = AN$.

COR. If L be the latus rectum, $L \times AN = NP^2$ (Prop. 2. Cor. 2): hence $L : NP :: NP : AN :: 2NP : 2AN (NT)$.

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PROP. VIII.

A perpendicular SY *from the focus upon any tangent* PT *intersects* PT *in the tangent at the vertex* A.



Draw the tangent AY, which is parallel to PN (Prop. 4. Cor.), and let it meet PT in Y: join SY.

Then $AN = AT$ (Prop. 7.), and $\therefore$ $PY = TY$ (Eucl. 2. VI.) Also $SP = ST$, and SY is common: Hence $\angle SYP = \angle SYT$, and SY is perpendicular to PT.

COR. The triangles SAY, SPY being similar,

$$SP : SY :: SY : SA,$$

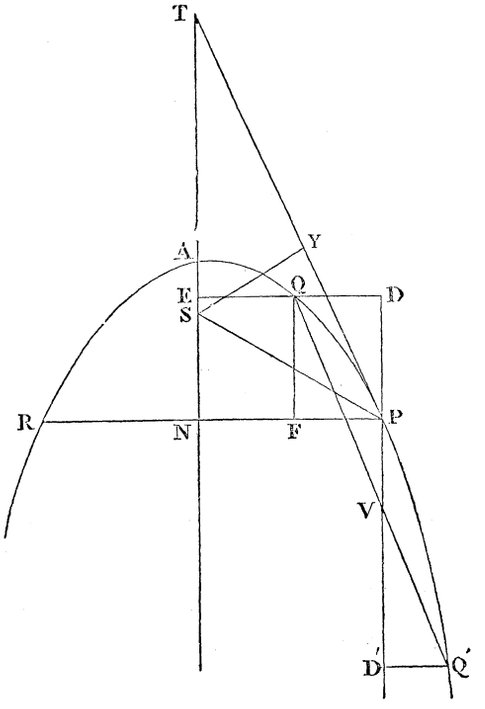
and $$SP : SA :: SP^2 : SY^2.$$

Also $SY^2 = SP \times SA$, and $\propto SP$.

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Prop. IX.

The rectangle by the latus rectum (L) *and the part* QF *of any diameter cut off by an ordinate to the axis, is equal to the rectangle by the segments* PF, FR, *of the ordinate.*



Draw QE perpendicular to the axis:

Then $PF \times FR = NP^2 - NF^2$ (Eucl. 5. II.)

$= NP^2 - QE^2 = L \times AN - L \times AE$ (Prop. 2. Cor. 2.)

$= L \times EN = L \times QF.$

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PROP. X.

If from either extremity of an ordinate QQ′ *to any diameter a perpendicular* QD *be let fall upon the diameter, the square of* QD *is equal to the rectangle by the latus rectum and the abscissa* PV.

The $\triangle^s$ QVD, PNT, SPY are manifestly similar;

Hence $QD : DV :: NP : NT$,

$:: L : PR$ (Prop. 7. Cor.)

$\therefore L \times DV = PR \times QD = PR \times PF$.

Also $L \times DP = L \times QF = RF \times PF$ (Prop. 9.)

Hence $L \times PV = PF^2$ (Eucl. 3. II.) $= QD^2$.

In the same manner if $Q'D'$ be the perpendicular from the other extremity Q', it may be shewn that $L \times PV = Q'D'^2$.

COR. $QD = Q'D'$, and the similar triangles QVD, $Q'VD'$ are therefore equal; hence $QV = Q'V$; *i. e.* a diameter bisects all its own ordinates.

PROP. XI.

The rectangle by four times the focal distance SP, *and the abscissa* PV *is equal to the square of the semi-ordinate* QV.

For $QV^2 : QD^2 :: SP^2 : SY^2$;

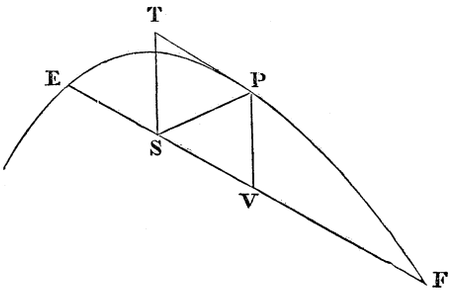
Or $QV^2 : L \times PV :: SP : SA$ (Prop. 8. Cor.)

$:: 4SP : L$;

$\therefore 4SP \times PV = QV^2$.

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Cor. 1. Let EF be the parameter. Then $SP = ST$ (Prop. 5.) $= PV$; also $EV^2 = 4SP \times PV = 4SP^2$.

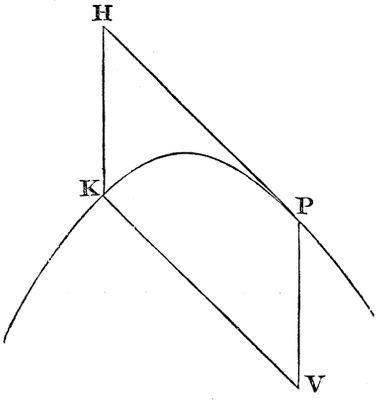


Hence $EV = 2SP$, and the parameter $= 4SP$.

Cor. 2. The rectangle by the parameter and abscissa of any diameter = the square of the corresponding semi-ordinate.

Cor. 3. The abscissa varies as the square of the semi-ordinate.

Cor. 4. If HK be drawn parallel to the axis

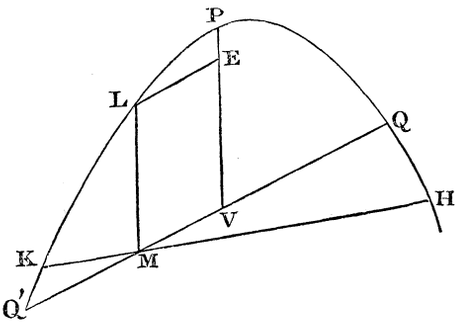


from any point H in a tangent, then $HK \propto PH^2$. For, if the parallelogram $PHKV$ be completed, $HK =$ the abscissa PV, and $PH =$ the semi-ordinate KV.

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Prop XII.

The rectangle by the parameter (P) *of any diameter* PV *and the part* LM *of any other diameter cut off by an ordinate* QQ′ *of the first, is equal to the rectangle by the segments* QM, MQ′ *of the ordinate.*



Draw LE parallel to QQ'.

Then $QM \times MQ' = QV^2 - VM^2$ (Eucl. 5. II.)

$= QV^2 - LE^2 = P \times PV - P \times PE$

(Prop. 11. Cor. 2.) $= P \times EV = P \times LM$.

Cor. 1. Let any other ordinate HK, whose parameter is P', intersect QQ' in M: then

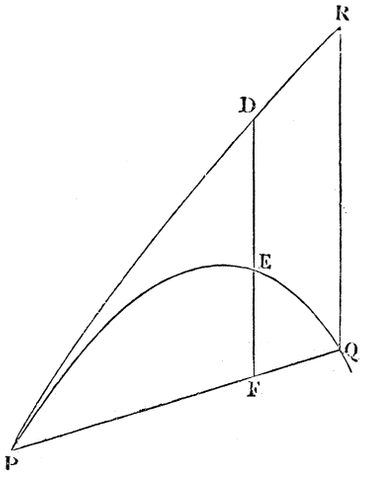
$$QM \times MQ' : HM \times MK :: P \times LM : P' \times LM$$
$$:: P : P'.$$

Cor. 2. If the lines QQ', HK, move parallel to themselves and intersect each other, either within or without the parabola, the rectangles of their segments are always in the same constant ratio.

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Prop. XIII.

If an ordinate PQ *and a tangent* PR *be drawn from the same point* P, *any diameter* DEF *terminated by the ordinate and tangent is divided by the curve in the same proportion in which itself divides the ordinate.*



Draw QR parallel to the axis;

Then $DE : QR :: PD^2 : PR^2$ (Prop. 11. Cor. 4.)
$:: PF^2 : PQ^2$,

And $QR : DF :: PQ : PF$;

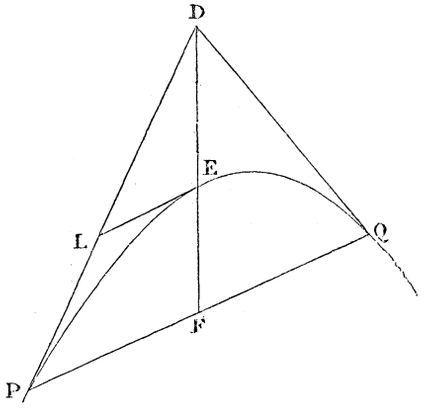
$\therefore DE : DF :: PF : PQ$,

Div^{do}. $DE : EF :: PF : FQ$.

Cor. 1. If DEF bisects PQ, it is then the diameter of which PQ is an ordinate, and DEF is the

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subtangent (Def. 6.): but then $DE = EF$, *i. e.* the



subtangent is double of the abscissa in all cases.

Cor. 2. If a tangent be drawn at Q, it must meet the diameter DEF in the same point D.

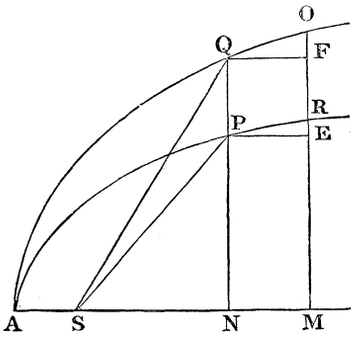
Cor. 3. The tangent at E is parallel to the ordinate PQ; hence $PL = LD$.

Prop. XIV.

If two parabolas, PAN, QAN, *whose latera recta are* L, L′, *respectively, have a common axis and vertex, the areas* PAN, QAN, *cut off by a common ordinate* QPN, *are in the subduplicate ratio of the latera recta.*

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Draw OM parallel to QN and PE, QF perpendicular



to OM, Then (Eucl. 1. VI.)

rectangle PM : rectangle QM :: PN : QN

$$:: \sqrt{L \times AN} : \sqrt{L' \times AN}$$

$$:: \sqrt{L} : \sqrt{L'};$$

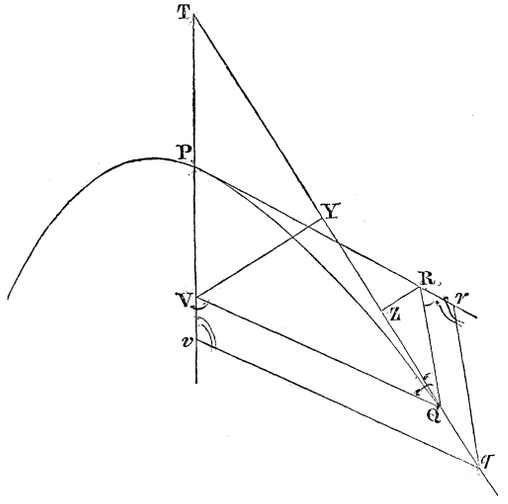
which constant ratio holds for all the corresponding rectangles thus inscribed in the areas PAN, QAN, and *componendo* for the sums of them in each, into how many parts soever the abscissa AM is divided. Let the number of the parts MN be increased, and the magnitude of each diminished, indefinitely. Then in the limit the sums of the inscribed rectangles are respectively equal to the areas PAN, QAN. (NEWTON. Lem. 4.) The areas therefore are in the constant proportion of $\sqrt{L}$ to $\sqrt{L'}$.

COR. Area ASP : area ASQ :: $\sqrt{L}$: $\sqrt{L'}$.

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Prop. XV.

The parabolic area, contained by the curve, the abscissa, and the semi-ordinate of any diameter, is two-thirds of the parallelogram completed of the abscissa and semi-ordinate.



Q, q, are points in the curve indefinitely near; PR, QT, tangents at P, Q; QR, qr, parallel to PV; VY, RZ perpendicular to QT.

Then the triangles TVY, RQZ being similar, and TV double of PV or QR, (Prop. 13. Cor. 1.) VY is also double of RZ.

But area $QVvq$: area $RQqr$ :: VY : RZ; that is, area $QVvq$ is double of $RQqr$.

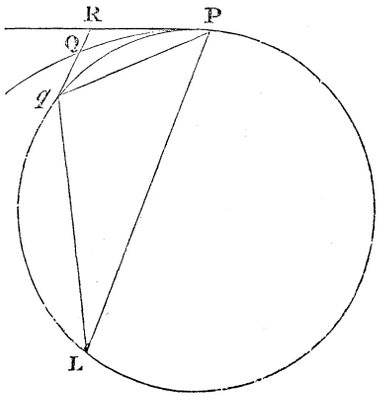
Hence the whole area PQV is double of the whole area PQR, and is therefore two thirds of the parallelogram VR.

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PROP. XVI.

To determine the diameter of the circle of curvature at any point of a parabola, and the chord which passes through the focus.

DEF. If a curve PQ and a circle PqL touch the



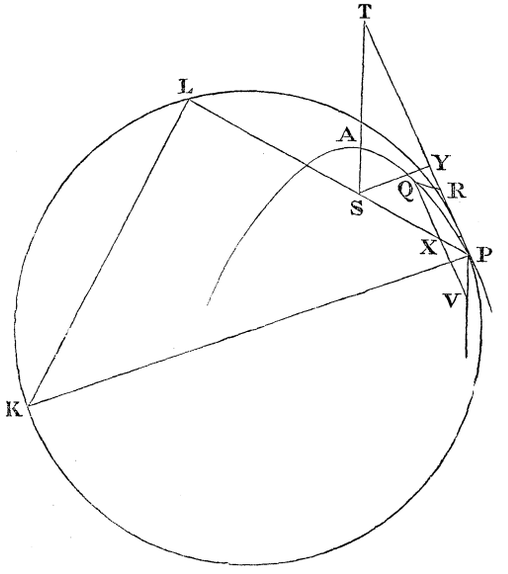
same straight line PR at the same point P, the circle is said to be a *Circle of Curvature* to the curve, when their deflections QR, qR from the common tangent PR are ultimately equal, or, which is the same thing, when indefinitely small arcs PQ, Pq, of the curve and circle, being equally deflected from the common tangent, are coincident.

COR. If any chord PL be drawn, and the subtense qQR parallel to PL, then the triangles PqL, PqR are equiangular (Eucl. 29. I. and 32. III.) Hence $PL = \frac{\overline{\text{chord. } Pq}^2}{qR}$. But in the evanescent state of the figure PqR, the arcs PQ, Pq are coincident and equal, and the subtenses QR, qr: also the chord Pq is then equal to the arc Pq (NEWT. Lem. 7.)

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Hence $PL = \frac{\overline{\text{arc } PQ}^2}{QR}$, PQ and QR being taken in the evanescent state.

Let now PLK be the circle of curvature at any



point P of the parabola, PL the chord through the focus S, PK the diameter. QR parallel to PL, and QV parallel to PT cutting SP in X. Then $QR = PX = PV$, because $SP = ST$ (Prop. 5.) Also QV is ultimately equal to the arc PQ. (NEWT. Lem. 7.)

Hence $PL = \frac{PQ^2}{QR} = \frac{QV^2}{QR} = \frac{4SP \times PV}{QR} = 4SP$.

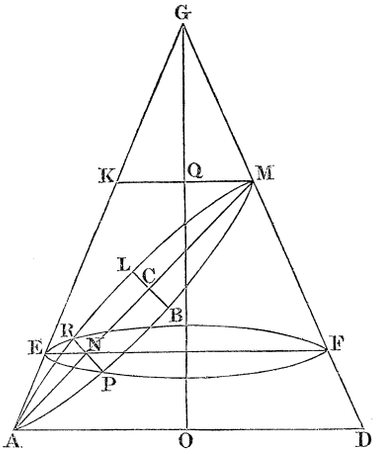
By sim. $\triangle^s$ SPY, SKL, the diameter $PK = \frac{PL \times SP}{SY} = \frac{4SP^2}{SY}$, or $= \frac{4SP^2}{\sqrt{SP \times SA}}$ (Prop. 8.) $= \frac{4SP^{\frac{3}{2}}}{SA^{\frac{1}{2}}}$, or $= \frac{4SP^3 \times SA}{SY \times SP \times SA} = \frac{L \times SP^3}{SY^3}$.

ON

The Ellipse.

DEFINITIONS.

1. If a right cone *GAD* be cut by a plane *APM* through both slant sides, the section is called an *Ellipse*.

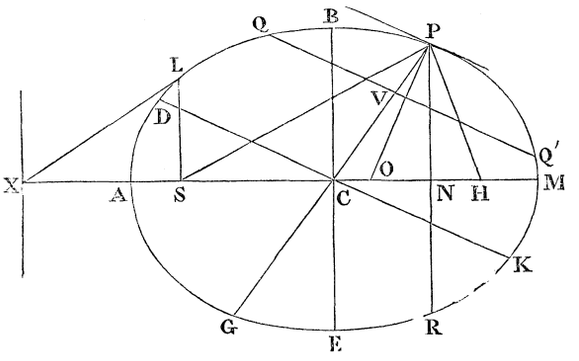


2. If *GAD* be that position of the revolving triangle which is perpendicular to the cutting plane *APM*, their common section *AM* is called the *Axis Major;* and the points *A*, *M*, are called its *Vertices:* also the point *C* which bisects *AM* is called the *Center*.

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3. A line *BCL* drawn through the center perpendicular to the axis major and terminated both ways by the curve, is called the *Axis Minor*.

4. If with the extremity *B* of the axis minor as center, and radius the semi-axis major, a circle be



described, cutting the axis major in the two points *S* and *H*, those two points are called the *Foci*.

5. A perpendicular *PNR* to the axis major, terminated by the curve, is called an *Ordinate* to the axis, and the segments *AN*, *NM*, into which it divides the axis, the *Abscissæ*. Also the ordinate through either focus is called the *Latus Rectum*.

6. Any line *PCG* through the center is called a *Diameter;* and a diameter *DCK* drawn parallel to the tangent at the extremity *P* of *PCG* is called the *Conjugate Diameter* to *PCG*.

7. A line *QQ'* drawn parallel to the tangent at any point *P* is called an *Ordinate* to the point *P* or diameter *PG*; and the segments *PV*, *VG*, of the diameter, the *Abscissæ*.

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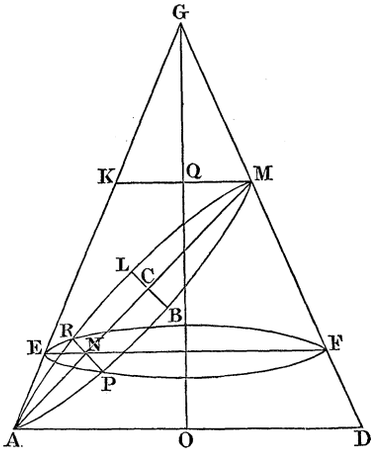
8. A perpendicular PO to the tangent at any point P, terminated by either axis, is called a *Normal;* and the part ON of the axis, the *Subnormal.*

9. If in the axis major produced a point X be taken, so that CX is a third proportional to CS and CA, a perpendicular to the axis major through that point X is called the *Directrix.*

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PROP. I.

The rectangle by the abscissæ AN, NM, *of the axis major is to the square of their semi-ordinate* NP *as the square of the axis major to the square of the axis minor.*



As in the Parabola, Prop. 1, PNR is perpendicular to AM and to EF, and is bisected by EF: the axis therefore bisects all its ordinates. Draw MQK parallel to AD.

Then $AN : EN :: AM : MK :: AC : MQ$

And $NM : NF :: AM : AD :: AC : AO$

$\therefore AN \times NM : EN \times NF (NP^2) :: AC^2 : AO \times MQ.$

When NP coincides with BC,

$$AC \times CM : BC^2 :: AC^2 : AO \times MQ$$

$\therefore BC^2 = AO \times MQ$;

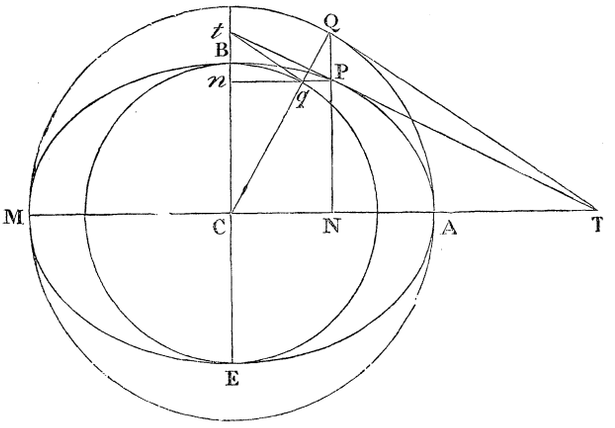
and $AN \times NM : NP^2 :: AC^2 : BC^2$.

COR. 1. $NP^2 \propto AN \times NM \propto CA^2 - CN^2$.

COR. 2. $NP^2 : CA^2 - CN^2 :: BC^2 : AC^2$.

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Cor. 3. If AQM be the circle on the axis major, and PN be produced to meet the circumference in Q; then, since $CA^2 - CN^2 = NQ^2$, $PN : QN :: BC : AC$.



Cor. 4. Let CQ cut the circle upon the axis minor in q; join Pq, and produce it to the axis minor in n. Then because $PN : QN :: Cq : CQ$ (Cor. 3.), Pqn is parallel to CN and perpendicular to BC.

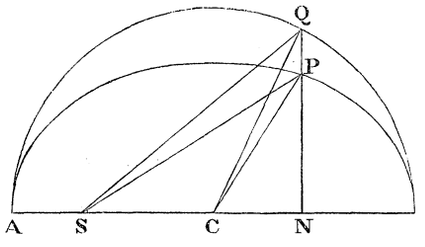
Hence $qn^2 : CN^2 :: Cn^2 (PN^2) : QN^2$ by sim. Δ^s

Or $Bn \times nE : Pn^2 :: BC^2 : AC^2$.

Cor. 5. $Bn \times nE \propto Pn^2$.

Cor. 6. $Pn : qn :: AC : BC$.

Cor. 7. As in the Parabola, Prop. XIV, the



area ANP of the ellipse : the area ANQ of the

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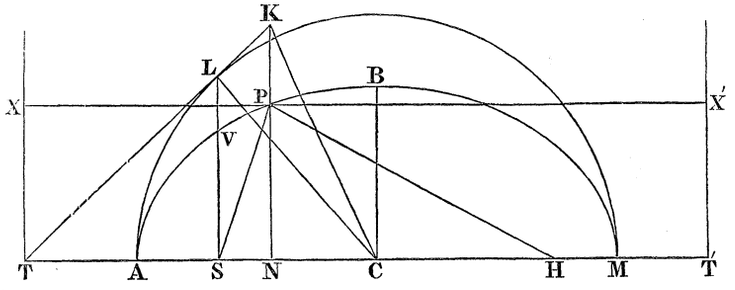
circle :: NP : NQ, *i. e.* :: BC : AC. The whole areas therefore of the ellipse and circle are in the same proportion. Hence the area of the ellipse = area of the circle $\times \frac{BC}{AC}$ which for different ellipses $\propto AC^2 \times \frac{BC}{AC} \propto AC \times BC \propto$ the rectangle by the axes.

Cor. 8. The areas of circles being as the squares of their diameters, the area of an ellipse = the area of that circle whose diameter is a mean proportional between the axes.

Cor. 9. Area ASP : area ASQ } :: BC : AC.
and area ACP : area ACQ }

Prop. II.

The distance SP *of any point* P *in the ellipse from the focus is to the perpendicular distance* PX *of the same point from the directrix in the constant ratio of* SC *to* AC.



Let SL, perpendicular to AM, meet the circle on the axis major in L; also let LT a tangent at L,

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meet the axis in T, and NP produced in K. Join CK, CL. · The perpendicular TX to the axis through T, is the directrix (Def. 9): also $SL = BC$, from Def. 4.

Then $NP^2 : CA^2 - CN^2 :: BC^2 : AC^2$ (Prop. 1. Cor. 2.)

And $SN^2 : KL^2 :: ST^2 : TL^2$

$:: SL^2 (BC^2) : CL^2 (AC^2)$ by sim. Δ^s.

$\therefore NP^2 + SN^2 (SP^2) : KL^2 + CA^2 - CN^2 :: BC^2 : AC^2$

But $KL^2 + CA^2 - CN^2 = CK^2 - CN^2 = KN^2$

Hence $SP : KN :: BC : AC$

And $KN : TN (PX) :: SC : SL (BC)$ by sim. Δ^s.

Therefore $SP : PX :: SC : CA$.

Prop. III.

The sum of the two focal distances SP, HP, *is equal to the axis major.*

Take $MT' = AT$; draw the directrix $T'X'$; and produce XP to meet $T'X'$ in X'.

Then $SP : PX$ and $HP : PX'$ $\Big\}$ $:: SC : AC$ (Prop. 2.) $:: AC : CT$

$\therefore SP + HP : PX + PX' (XX') :: AC : CT$

But $XX' = TT' = 2CT$; $\therefore SP + HP = 2AC$.

Cor. 1. $HP = 2AC - SP$.

Cor. 2. $AS \times SM = SL^2 = BC^2$.

Cor. 3. Let SV be the semi latus rectum: then $SV : SL (BC) :: BC : AC$ (Prop. 1. Cor. 3.) that is, the latus rectum is a third proportional to the axis major and axis minor.

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Prop. IV.

The focal distance $\mathrm{SP} = \dfrac{\mathrm{BC}^2}{\mathrm{AC} - \mathrm{SC} \times \text{cos. PSN}}$, *radius being unity.*

$(2AC - SP)^2 = HP^2$ (Prop. 3. Cor. 1.)

$= SP^2 + SH^2 - 2SH \times SN$ (Eucl. 13. II.)

Or $4AC^2 - 4AC \times SP + SP^2 = SP^2 + 4SC^2 - 4SC.SN$

Hence $AC \times SP - SC \times SN = AC^2 - SC^2 = BC^2$.

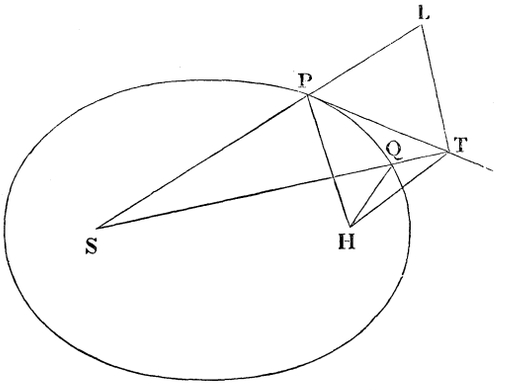
But $SN = SP \times \text{cos. } PSN$, rad = 1,

$\therefore\ AC \times SP - SC \times SP \times \text{cos. } PSN = BC^2$,

And $SP = \dfrac{BC^2}{AC - SC \times \text{cos. } PSN}$.

Prop. V.

If one of the focal distances SP *be produced, the line* PT *which bisects the exterior angle* HPL, *touches the curve at* P.



In SP produced take $PL = PH$, and in PT take any point T; join ST meeting the ellipse in Q: join also LT, HT, HQ.

Then $LT = HT$ (Eucl. 4. I.); also $ST + HT = ST + TL$, and is therefore greater than SL, or than

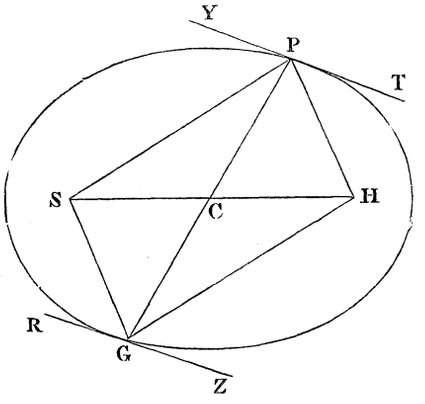
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$SP+HP$ or than $SQ+HQ$. Hence the point Q lies between S and T (Eucl. 21. 1.), or T is without the ellipse; and since every point of PT, except P, is without the ellipse, PT touches at P.

Cor. 1. SP, HP make equal angles with the tangent PT.

Cor. 2. A tangent at the extremity of either axi' is perpendicular to that axis.

Cor. 3. Complete the parallelogram $SPHG$.



Then $SG+GH=SP+PH$ (Eucl. 34. 1.), and G is a point in the ellipse. Join CP, CG; then because the diagonals of parallelograms bisect each other, and that $SC=CH$, PCG is a straight line and a diameter; *i. e.* the center bisects all diameters.

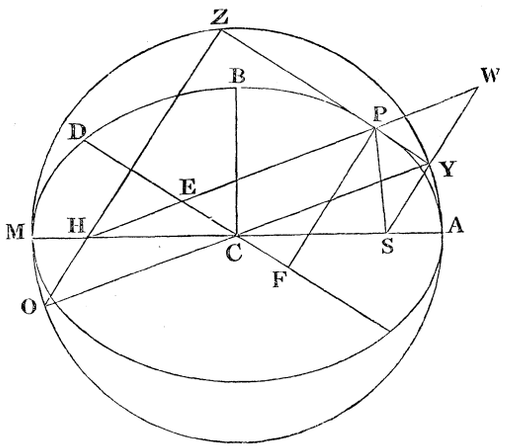
Cor. 4. Draw the tangents YT, RZ, at P, G: Then $\angle SPY=\angle HPT$ (Cor. 1.) and $\therefore = \frac{1}{2}$ supp^t. of $SPH=\frac{1}{2}$ supplement of $SGH=\angle HGZ$; and $\angle SPG=$ alternate $\angle PGH$; $\therefore$ $\angle YPG=\angle PGZ$, and YT is parallel to RZ.

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Prop. VI.

The perpendiculars from the foci upon any tangent intersect the tangent in the circumference of a circle whose diameter is the axis major.

Produce HP to W, making $PW = SP$; let SW cut the tangent in Y, and join CY.



The Δ^s SPY, WPY are equal in all respects (Eucl. 4. 1.), whence $\angle SYP = \angle WYP$, and SY is perpendicular to PY. Also $SY = YW$, and $CS = CH$, therefore CY is parallel to HW, and by sim. Δ^s, $CY = \frac{1}{2} HW = \frac{1}{2} (SP + HP) = CA$; therefore Y is a point in the circumference of the circle whose radius is CA. In like manner, Z is a point in the circumference of the same circle.

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Cor. Let a conjugate diameter CD cut either focal distance HP in E. Then by the proposition, CY is parallel to HP, and $PECY$ is a parallelogram. Hence $PE = CY = CA$.

Prop. VII.

The rectangle by the perpendiculars from the foci upon the tangent is equal to the square of half the minor axis.

Produce ZH to the circumference in O; join CO. Then $\angle OZY$ being a right angle is in a semicircle, and O, Y, are the extremities of a diameter: OCY is therefore a straight line and a diameter; also the $\triangle^s$ OHC, YSC are similar and equal, and $SY = OH$; $\therefore SY \times HZ = OH \times HZ = AH \times HM$ (Eucl. 35. III.) $= BC^2$ (Prop. 3. Cor. 2.)

Cor. Since $\angle SPY = \angle HPZ$ (Prop. 5, Cor. 1.), the $\triangle^s$ SPY, HPZ are similar.

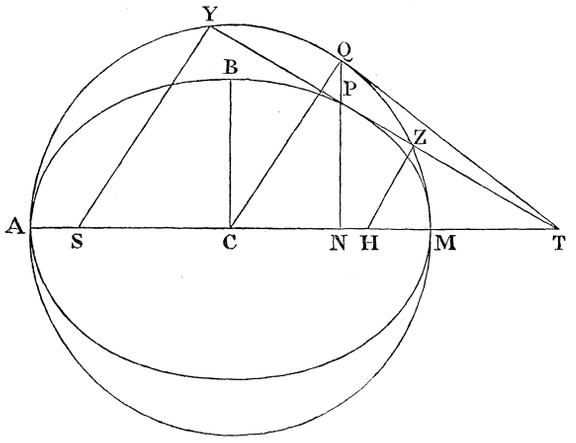
Hence $SY = HZ \times \frac{SP}{HP}$, and $SY^2 = BC^2 \times \frac{SP}{HP}$.

Also $SY^2 \propto \frac{SP}{HP}$ or $\propto \frac{SP}{2AC - SP}$.

Prop. VIII.

The semi-axis major is a mean proportional between the distance CN *of any ordinate from the center and the distance* CT *of the intersection of the tangent at* P *with the axis.*

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Produce NP to Q; and join TQ, CQ

Then $TY : SY$
And $TZ : HZ$ $\Big\}$ :: $TN : PN$

$\therefore TY \times TZ (MT \times TA) : SY \times HZ (BC^2) :: TN^2 : PN^2$

And $BC^2 : AC^2 :: PN^2 : QN^2$.

Hence $MT \times TA : CA^2 :: TN^2 : QN^2$

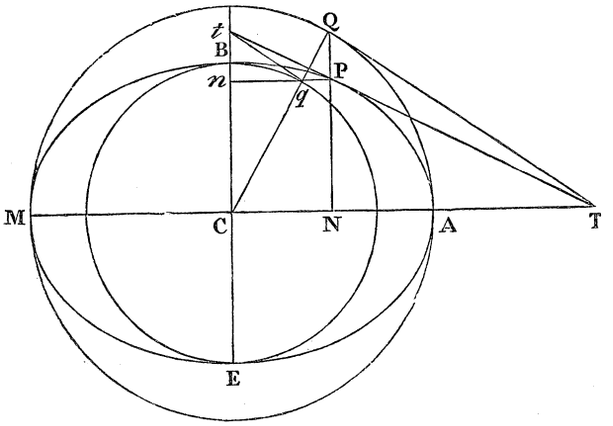
Comp^{do}. $CT^2 : CA^2 (CQ^2) :: QT^2 : QN^2$.

The triangles CQT, TQN have therefore a common angle CTQ, the sides about two other angles proportional, and the third angle TNQ in one a right angle; they are therefore similar (Euc. 7. VI.) Hence $\angle CQT$ is a right angle, and TQ touches at Q; *i. e.* the tangents at P and Q intersect the axis in the same point T.

Also $CN : CQ (CA) :: CQ (CA) : CT$.

Cor. Produce the tangent TP to meet the axis minor in t; and join qt.

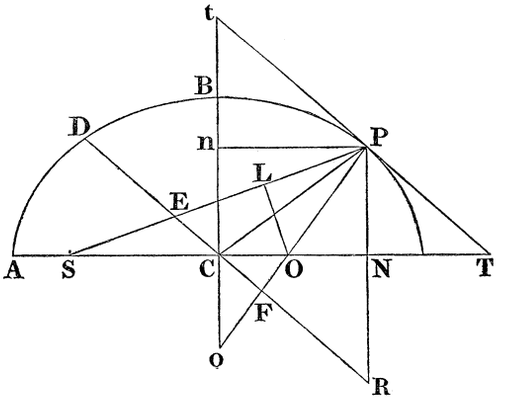
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Then $Ct : Cn :: CT : NT :: CT^2 : QT^2$
$:: Cq^2$ or $BC^2 : Cn^2$
Hence $Ct : BC :: BC : Cn$, as for the axis major.

Prop. IX.

The rectangle by the normal PO *to either axis and the perpendicular* POF *upon the conjugate diameter is equal to the square of half the other axis.*



Let PN produced meet the conjugate in R. Then the angles N and F being right angles, a circle

would circumscribe the quadrilateral $FONR$; hence $PO \times PF = PN \times PR$ (Eucl. 36. III.) $= Cn \times Ct = BC^2$ (Prop. 8. Cor.)

Similarly $Po \times PF = CN \times CT = AC^2$.

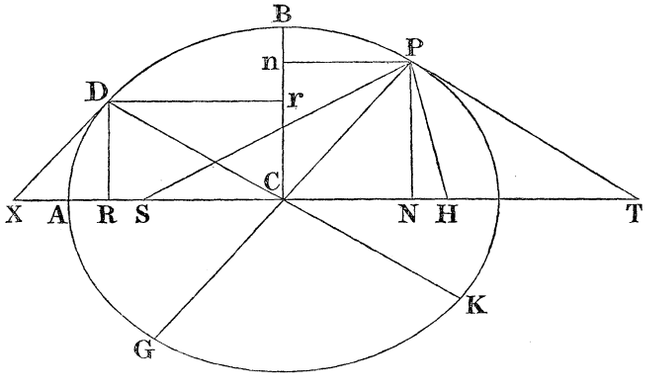
Cor. 1. The triangles PON, Pon, being similar, $ON : Pn\,(CN) :: PO : Po :: BC^2 : AC^2 :: L : 2AC$ (Prop. 3. Cor. 3.)

Hence the subnormal $ON = \frac{L}{2AC} \times CN$, L being the latus rectum. Similarly $on = \frac{2AC}{L} \times Cn$.

Cor. 2. From O draw OL perpendicular to SP; then from the quadrilateral $EFOL$, whose opposite angles F and L are right angles, $PL = \frac{PF \times PO}{PE} = \frac{BC^2}{AC}$ = half the latus rectum.

Prop. X.

If one diameter DCK *be conjugate to another* PCG, *conversely* PCG *is conjugate to* DCK.



Draw PN, DR, ordinates to the axis, and the tangents PT, DX.

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Then $CN \times CT = CA^2$ (Prop. 8.) Take away CN^2, and $CN \times NT = CA^2 - CN^2$ (Eucl. 3. II.) $= AN \times NM$.
$\therefore CN \times NT : CR \times RX :: PN^2 : DR^2$ (Prop. 1. Cor. 1.)
$:: NT^2 : CR^2$, by sim. $\triangle^s$.
Hence $CN : RX :: NT : CR :: PN : DR$;
$\therefore \triangle^s CPN, DRX$ are similar (Eucl. 6. VI.) and PC is parallel to DX.

PROP. XI.

The distance of one of the ordinates PN, DR *from the center is a mean proportional between the distances of the other ordinate from the center and from the intersection of its tangent with the axis.*

$CN \times CT = CR \times CX$, for each $= CA^2$ (Prop. 8.)
Hence $CN : CR :: CX : CT$
$:: CD : PT$ by sim. $\triangle^s CPT, CDX$
$:: CR : NT$ by sim. $\triangle^s CDR, PNT$.
So $CR : CN :: CN : RX$.

COR. 1. $CN \times NT\,(CR^2) = CA^2 - CN^2$ (Prop. 10.)
Hence $CA^2 = CN^2 + CR^2$. By a similar proof, if Pn, Dr, be perpendicular to the axis minor, $Cn^2 + Cr^2 = BC^2$. Wherefore $AC^2 + BC^2 = CN^2 + Cn^2 + CR^2 + Cr^2 = CP^2 + CD^2$.

COR. 2. $SP^2 + HP^2 + 2SP \times HP = 4AC^2$.
But $SP^2 + HP^2 = 2SC^2 + 2CP^2$ (Eucl. 12 and 13. II.)
$\therefore SP \times HP = 2AC^2 - SC^2 - CP^2 = AC^2 + BC^2 - CP^2 = CD^2$, by the preceding Corollary.

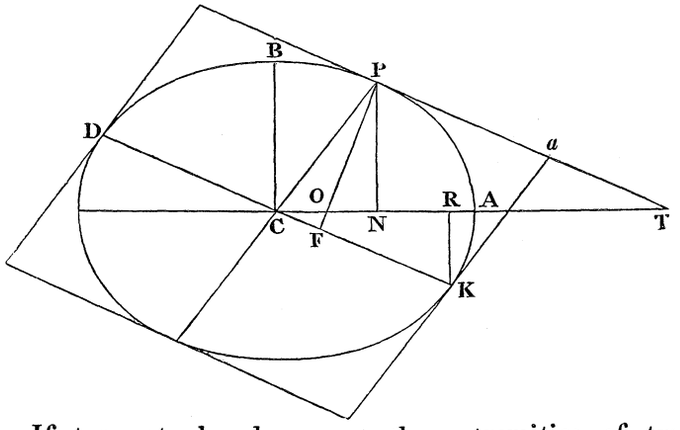
COR. 3. $NP^2 : AN \times NM\,(CR^2) :: BC^2 : AC^2$
$\therefore NP : CR :: BC : AC$
So $DR : CN :: BC : AC$.

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Cor. 4. $NP : CR :: DR : CN$. The $\triangle^s$ CPN, CDR, are therefore equal. (Eucl. 15. vi.)

Prop. XII.

All parallelograms, circumscribing an ellipse at the extremities of conjugate diameters, are equal.



If tangents be drawn at the extremities of two conjugate diameters, they form a parallelogram circumscribing the ellipse (Prop. 5. Cor. 4. and Prop. 10.) of which $CPaK$ is a fourth part. Draw POF perpendicular to DCK. Then by sim. $\triangle^s$ PON, CKR

$CK(CD) : PO :: CR : PN :: AC : BC$ (Prop. 11. Cor. 3.)

$\therefore CD \times PF : PO \times PF (BC^2) :: AC \times BC : BC^2$ (Prop. 9.)

Hence $CD \times PF = AC \times BC$, or the parallelogram PK = the rectangle by the semi-axes.

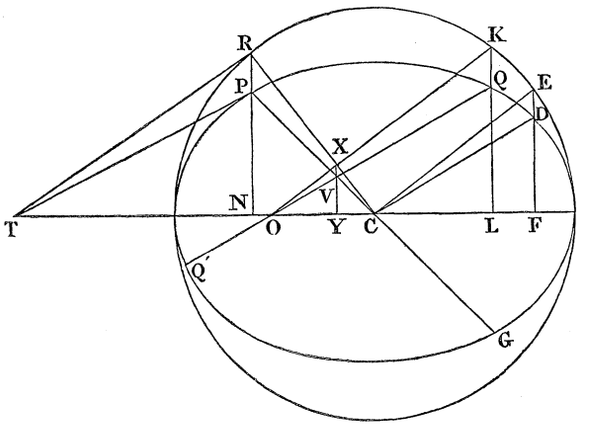
Cor. The whole area of the ellipse, which varies as $AC \times BC$ (Prop. 1. Cor. 6.), varies also as $CD \times PF$, or as the circumscribing parallelogram.

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Prop. XIII.

The rectangle by the abscissæ PV, VG, *of any diameter is to the square of their semi-ordinate* QV *as the square of the semi-diameter* CP *to the square of the semi-conjugate* CD.

Let the ordinate QQ' meet the axis in O: draw



RPN, VY, QL, EDF, perpendicular to the axis: let YV meet CR in X, and OX meet LQ in K. Then since $KL : QL :: XY : VY :: RN : PN$, by similar triangles, K is in the circumference of the circle (Prop. 1. Cor. 3.) But CD is parallel to PT'; $\therefore TN : CF :: PN : DF :: RN : EF$; whence the $\triangle^s$ $T'NR$, CFE, are similar, and EC is parallel to RT' and perpendicular to CR. In like manner KO is perpendicular to CR.

Now $CP^2 : CV^2 :: CR^2 : CX^2$

$\therefore CP^2 - CV^2 : CR^2 - CX^2 (KX^2) :: CP^2 : CR^2$

And $KX^2 : QV^2 :: CE^2 : CD^2$

$\therefore CP^2 - CV^2 (PV \times VG) : QV^2 :: CP^2 : CD^2$.

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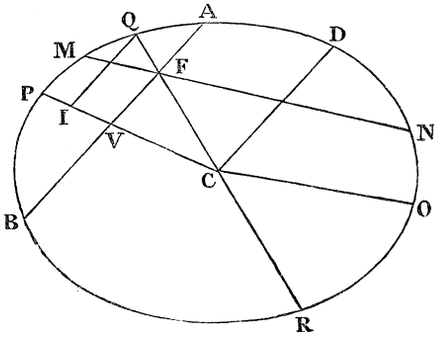
Cor. 1. $PV \times VG : QV^2 :: CP^2 : CD^2$, by the same proof; $\therefore QV = QV$; *i.e.* a diameter bisects all its own ordinates.

Cor. 2. $PV \times VG$ or $CP^2 - CV^2 \propto QV^2$.

Cor. 3. Since $CP^2 - CV^2 : CP^2 :: QV^2 : CD^2$,
Div^do. $CV^2 : CP^2 :: CD^2 - QV^2 : CD^2$,
Whence $CD^2 - QV^2 \propto CV^2$.

Prop. XIV.

If any line AB *intersect a diameter* QCR *in* F, *and* CD *be parallel to* AB, *then the rectangle* AF, FB : *the rectangle* QF, FR :: $CD^2 : CQ^2$.



Draw CP conjugate to CD and therefore bisecting AB in V: also draw QI parallel to AB.

Then $CD^2 - QI^2 : CD^2 - AV^2 :: CI^2 : CV^2$ (Pr. 13. Cor. 3.)

Div^do. $AV^2 - QI^2 : CD^2 - QI^2 :: CI^2 - CV^2 : CI^2$
$:: QI^2 - FV^2 : QI^2$ by sim. $\triangle^s$.

Hence $AV^2 - FV^2 : CD^2 :: QI^2 - FV^2 : QI^2$ (Eu. 12. v.)
$:: CQ^2 - CF^2 : CQ^2$ by sim. $\triangle^s$.

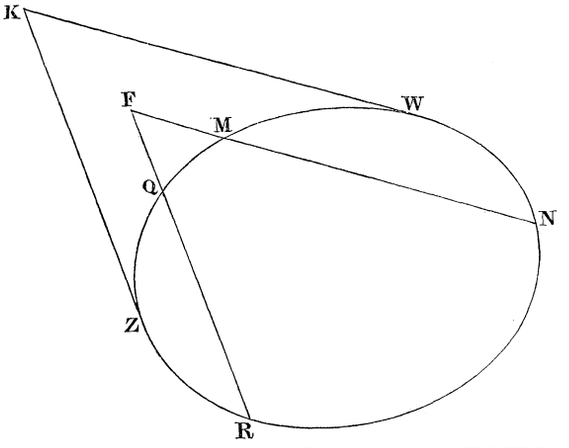
Or $AF \times FB : CD^2 :: QF \times FR : CQ^2$.

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Cor. 1. If any two lines whatever AB, MN intersect in F, the rectangles $AF \times FB$, $MF \times FN$, are as the squares of the diameters CD, CO, drawn parallel to AB, MN, respectively.

Cor. 2 If AB, MN, move parallel to themselves, and intersect any where either within or without the ellipse in F, the rectangles $AF \times FB$, $MF \times FN$, have always the same constant ratio.

Cor. 3. If from any point K without the ellipse tangents KW, KZ be drawn, then are KW and KZ in the same proportion as the diameters to which they are respectively parallel.



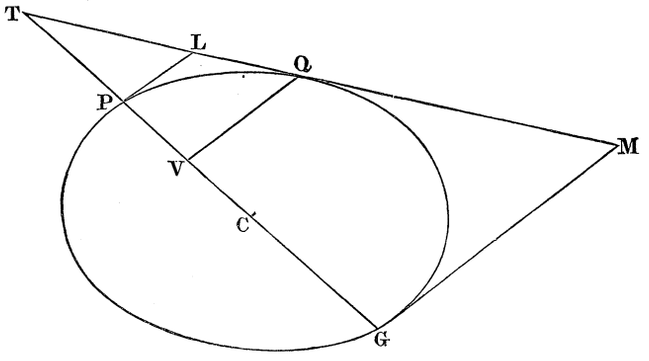
For take any other point F, and draw FMN, FQR parallel to KW, KZ. Then if FMN move parallel to itself until it coincides with KW, the rectangle FM, FN, becomes the square of KW at the point of contact: and so the rectangle FQ, FR becomes the square of KZ. But $FM \times FN$ to $FQ \times FR$, and therefore KW^2 to KZ^2, is as the squares of the diameters to which they are parallel.

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PROP. XV.

If a tangent QT *drawn at the extremity* Q *of any ordinate* QV *meet the diameter* CP *to which the ordinate belongs in* T; *then* CP *is a mean proportional between* CV *and* CT.

At the extremities, P, G of the diameter draw PL, GM touching the ellipse, and meeting the tangent QT in L and M.



Then $MQ : MG :: LQ : LP$ (Prop. 14. Cor. 3.)

And $MQ : LQ :: MG : LP$

Whence $VG : PV :: TG : TP$ by sim. triangles

$\therefore VG - PV : VG + PV :: TG - TP : TG + TP$

Or $2CV : 2CP :: 2CP : 2CT$.

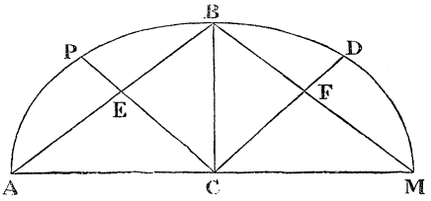
COR. 1. A tangent at the other extremity of the ordinate would meet the diameter CP produced at the same point T.

COR. 2. $PV \times VG = CP^2 - CV^2 = CV \times CT - CV^2 = CV \times VT$.

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Prop. XVI.

The diameters which bisect the lines joining the extremities of the axes are equal and conjugate.



Let AB, BM be bisected in E, F: then the Δ^s BCE, BCF are manifestly equal and similar, and $\angle ECB = \angle FCB$; therefore CP and CD making equal angles with CB are similarly drawn in the two elliptic quadrants and consequently are equal. Also AB is bisected by CP and is therefore an ordinate belonging to the diameter CP (Prop. 13. Cor. 1.) But $AC = CM$, and $BF = FM$; hence CFD is parallel to AB or is conjugate to CP.

Prop. XVII.

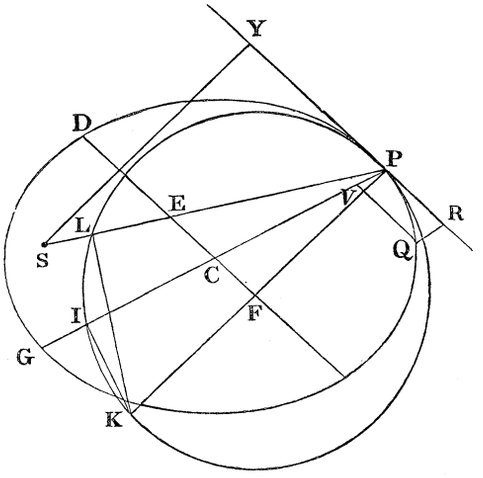
To determine the diameter of curvature and the chords through the center and focus of an ellipse.

1. The chord PI through the center.

Draw QR parallel to PI and QV parallel to the tangent; then when PV is evanescent, $PQ = QV$.

Now $PV \times VG : QV^2 :: CP^2 : CD^2$ (Prop. 13.); hence $\frac{QV^2}{PV} = VG \times \frac{CD^2}{CP^2}$; and ultimately $\frac{PQ^2}{QR} = \frac{2CP \times CD^2}{CP^2} = \frac{2CD^2}{CP} = PI$.

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II. The diameter PK.

In the quadrilateral $CFKI$, the opposite angles F and I are right angles, and therefore a circle would circumscribe it. Hence $PK \times PF = CP \times PI = 2CD^2$, by the preceding case, and $PK = \frac{2CD^2}{PF}$.

III. The chord PL through the focus.

Let it cut the conjugate in E; then PE = the semi-axis major. Also, as in the last case, $PL \times PE = PK \times PF = 2CD^2$; $\therefore PL = \frac{2CD^2}{PE}$.

Cor. The diameter $PK = \frac{2CD^2}{PF} = \frac{2PE^2 \times BC^2}{PF^3}$ (Prop. 12.) $= \frac{2BC^2}{PE} \times \frac{PE^3}{PF^3} = L \times \frac{SP^3}{SY^3}$ by similar triangles.

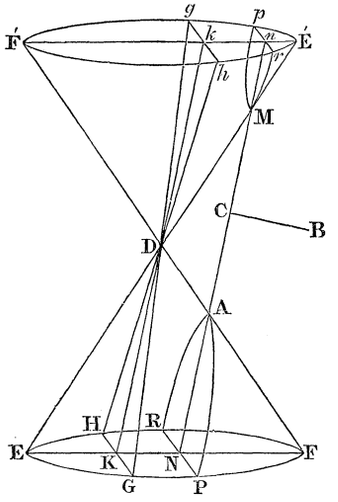
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ON

The Hyperbola.

DEFINITIONS.

1. If a right cone *DEF*, and another *DE'F'* which is equal and vertical to *DEF*, be cut by a plane *RAMrp*, each of the sections *RAP*, *Mpr*, is called an *Hyperbola*, and the two are called *Opposite Hyperbolas*.



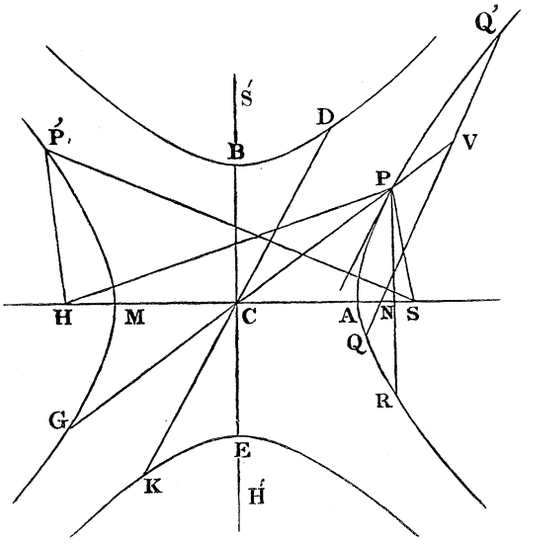
Scholium. Every section *DGH* of the cone passing through the vertex *D* is a triangle, because the slant side must coincide with the hypothenuse of the revolving triangle in some one position of it.

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2. The interval AM between the opposite hyperbolas is called the *Axis Major*, and the points A, M, are called *Vertices:* also the point C which bisects AM is called the *Center.*

3. If $DHGgh$ be the triangular section through the vertex D whose plane is parallel to the cutting plane; and BC, perpendicular to AM at C, be taken a fourth proportional to DK, KG, and AC; then BC is called the *Axis Minor* or *Conjugate Axis;* which is manifestly the same for both RAP and Mrp.

4. The two opposite hyperbolas BD, EK, whose axis major is BE and conjugate axis AM, are called the *Conjugate Hyperbolas* to AP, MP'.



5. If with center C, and radius CS equal to the line joining the vertices A, B, of the major and minor

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axis, a circle be described cutting the axes produced in S, S', H, H', respectively, those four points are called the *Foci*.

6. If the axes AM, BE be equal, the curve is called a *Rectangular Hyperbola*. In this case the conjugate hyperbolas AP, BD, are equal and similar.

7. Any line PCG drawn through the center and terminated by two opposite hyperbolas is called a *Diameter*, and the intersections of a diameter with the curves are called it's *Vertices*.

8. A perpendicular PNR to the axis major, terminated by the curve, is called an *Ordinate* to the axis; and the distances AN, NM of the ordinate from the vertices are called the *Abscissæ*.

9. A line QQ' drawn parallel to the tangent at any point P and terminated by the curve, is called an *Ordinate* to the point P or diameter PCG; and the distances PV, VG of the ordinate from the vertices of the diameter are called the *Abscissæ*.

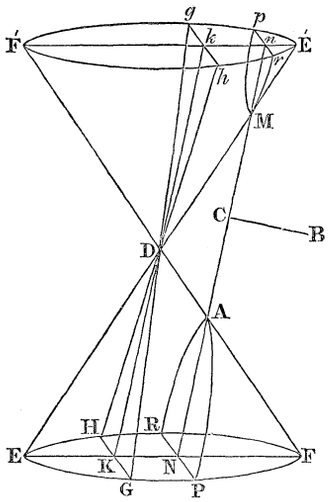
10. The *Latus Rectum*, *Normal*, *Subnormal*, *Conjugate Diameter*, and *Directrix*, are defined as in the Ellipse.

11. An *Asymptote* is a straight line which approaches nearer to meet a curve, the farther it is produced, but which being produced ever so far does never actually meet it.

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Prop. I.

The rectangle by the abscissæ AN, NM *of the axis is to the square of their semi-ordinate* NP *as the square of the axis major to the square of the axis minor.*



Let DGH be the triangular section through D whose plane is parallel to RAP. Then as in the Ellipse, Prop. 1, PNR is perpendicular both to EF and AN, and is bisected by EF: the axis therefore bisects all its ordinates. Also GH is perpendicular to EF, which bisects it in K.

Then $AN : NF :: DK : FK$

And $NM : EN :: DK : EK$

$\therefore AN \times NM : EN \times NF (NP^2) :: DK^2 : EK \times FK (KG^2)$

$:: AC^2 : BC^2$ (Def. 3.)

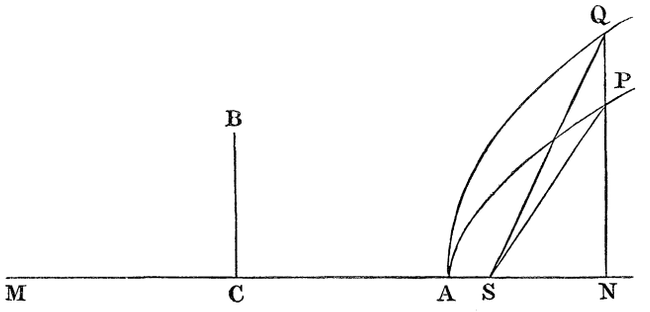
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Cor. 1. $PN^2 : CN^2 - CA^2 :: BC^2 : AC^2$.

Cor. 2. $PN^2 \propto AN \times NM \propto CN^2 - CA^2$.

Cor. 3. In the rectangular hyperbola, the rectangle of the abscissæ is equal to the square of the corresponding semi-ordinate. (Def. 6.)

Cor. 4. If AQ be a rectangular hyberbola, AP any



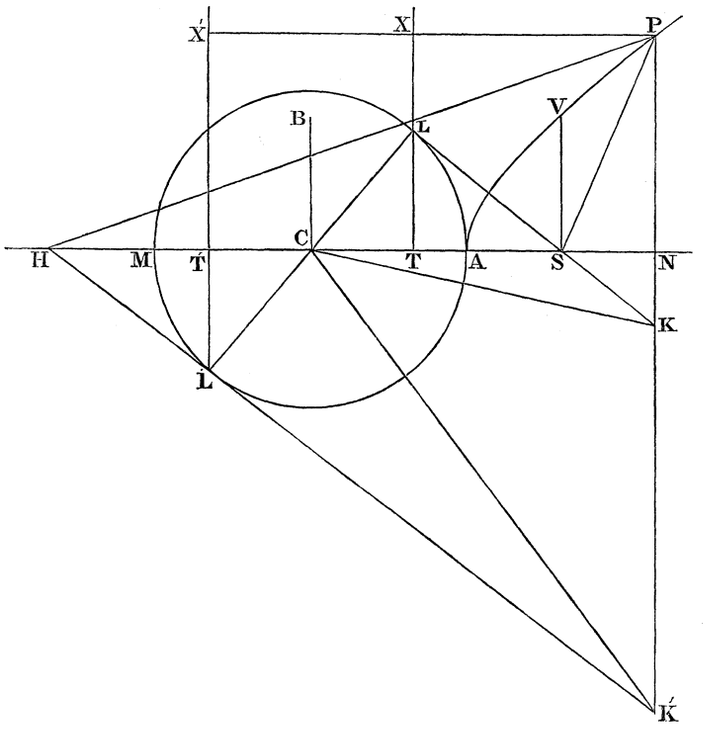
other hyperbola having the same axis major, and QPN a common ordinate of the axis, then $PN : QN :: BC : AC$. Hence as in the Parabola, Prop. XIV, the areas APN, AQN, as also the areas ASP, ASQ, are in the same constant proportion of BC to AC.

Prop. II.

The distance SP *of any point* P *in the hyperbola from the focus is to the perpendicular distance* PX *of the same point from the directrix in the constant ratio of* SC *to* CA.

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Let SL touch the circle upon the axis major at



L; then $SL = BC$, from Def. 5; and XLT perpendicular to AM through L is the directrix. Produce LS to meet PN produced in K; and join CK.

Then $PN^2 : CN^2 - CA^2 :: BC^2 : AC^2$ (Prop. 1. Cor. 1.)

Also $SN^2 : KN^2 :: LS^2 (BC^2) : CL^2 (AC^2)$ by sim. $\triangle^s$.

$\therefore PN^2 + SN^2 (SP^2) : KN^2 + CN^2 - CA^2 :: BC^2 : AC^2$

But $KN^2 + CN^2 - CA^2 = CK^2 - CL^2 = KL^2$.

Hence $SP : KL :: BC : AC$

And $KL : TN (PX) :: SL : ST :: CS : SL (BC)$ by sim. $\triangle^s$. Therefore $SP : PX :: SC : AC$.

By the same proof, if LC be produced to L', and the tangent HL' be produced to meet PN in K'; also the directrix $L'T''X'$, and PX' perpendicular to it, be drawn, and CK' joined,

$$HP : PX' :: CS : CA.$$

PROP. III.

The difference of the focal distances SP, HP *is equal to the axis major.*

For by Prop. 2, $\left.\begin{matrix} SP : PX \\ \text{and } HP : PX' \end{matrix}\right\} :: SC : AC :: AC : CT.$

Hence $HP - SP : PX' - PX (XX') :: AC : CT$

But $XX' = TT' = 2CT$; therefore $HP - SP = 2AC$.

COR. 1. $HP = 2AC + SP$.

COR. 2. $AS \times SM = SL^2$ (Eucl. 36. III.) $= BC^2$.

COR. 3. If SV be the semi-latus rectum,

$AC^2 : BC^2 :: AS \times SM (BC^2) : SV^2$ (Prop. 1.)

The latus rectum is therefore a third proportional to the axis major and minor.

PROP. IV.

The focal distance $\mathrm{SP} = \dfrac{\mathrm{BC}^2}{\mathrm{AC} - \mathrm{SC} \times \textit{cos.}\ \mathrm{PSN}}$, *radius being unity.*

For since $(2AC + SP)^2 = HP^2$, (Prop. 3. Cor. 1.)

$4AC^2 + 4AC \times SP + SP^2 = SP^2 + SH^2 + 2SH \times SN$

Whence $AC \times SP - SC \times SN = SC^2 - AC^2 = BC^2$

Or, since $SN = SP \times \cos. PSN$,

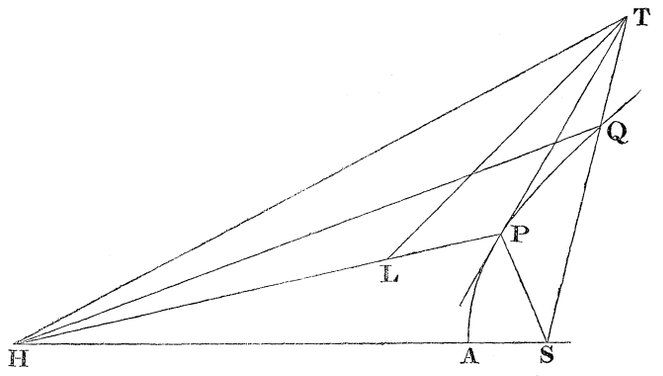
$AC \times SP - SC \times SP \times \cos. PSN = BC^2$,

And $SP = \dfrac{BC^2}{AC - SC \times \cos. PSN}$.

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PROP. V.

The line PT, *which bisects the angle* SPH, *touches the hyperbola at* P.



From PH cut off $PL=SP$, and in PT take any point whatever T. Join HT, LT, ST; then $LT=ST$ (Eucl. 4. 1.) In ST take any point Q nearer to S than T is, and join HQ.

Then HT is less than $HQ+QT$ (Eucl. 20. 1.) and taking away ST, $HT-ST$ is less than $HQ-SQ$. The nearer therefore to S the point Q is taken, the greater is the difference $HQ-SQ$, which in the limit $=HS$, when Q coincides with S.

But since HT is less than $HL+LT$, taking away LT, $HT-LT$ or $HT-ST$ is less than HL or $HP-SP$. It follows, that there is some point Q between T and S, where $HQ-SQ=HP-SP$, and

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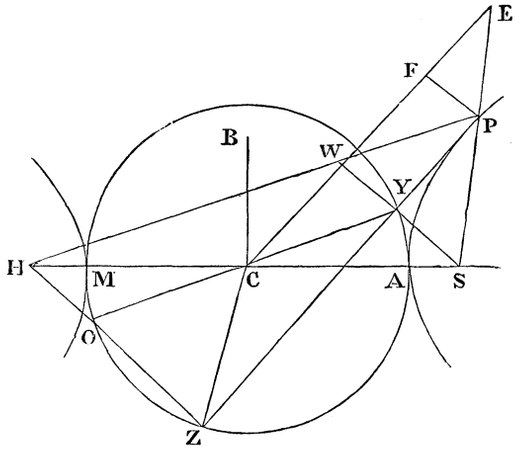
where consequently the curve cuts ST. The point T is therefore without the hyperbola, and the same being true of every point in PT, except P, PT touches the curve at P.

Cor. 1. SP and HP make equal angles with every tangent PT.

Cor. 2. A tangent at the vertex of either axis is perpendicular to that axis.

Prop. VI.

The perpendiculars from the foci on any tangent intersect the tangent in the circumference of a circle whose diameter is the axis major.



In PH take $PW=SP$, and then $HW=AM$; let SW cut the tangent in Y, and join CY.

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Then the triangles SYP, WYP are equal in all respects; $\therefore \angle SYP = \angle WYP$, and SY is perpendicular to PY. Also $SY = YW$, and $SC = CH$; $\therefore CY$ is parallel to HW, and $= \frac{1}{2} HW = CA$; *i. e.* Y is a point in the circumference. In like manner, if HZ be perpendicular to the tangent, Z is also in the circumference.

Cor. Join CZ which is parallel to SP, as in the proposition; and let SP produced meet the conjugate diameter in E. Then $PECZ$ is a parallelogram, and $PE = CZ = AC$.

Prop. VII.

The rectangle by the perpendiculars from the foci on any tangent is equal to the square of half the minor axis.

Let HZ meet the circle again in O, and join CO; then the right angle YZO is in a semicircle; hence YCO is a diameter and a straight line; also the triangles CSY, CHO are equal in all respects; hence $HO = SY$, and $SY \times HZ = HO \times HZ = HA \times HM$ (Eucl. 36. III.) $= BC^2$ (Prop. 3. Cor. 2.)

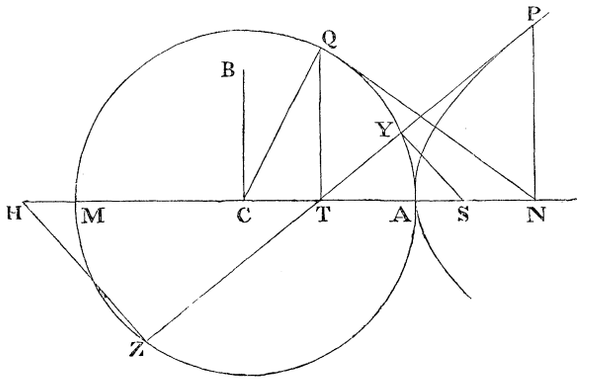
Cor. $SY^2 = BC^2 \times \frac{SP}{HP}$, as in the Ellipse, and

$$SY^2 \propto \frac{SP}{HP} \propto \frac{SP}{2AC + SP}.$$

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Prop. VIII.

The semi-axis major is a mean proportional between the distance CN *of any ordinate to the axis from the center and the distance* CT *of the intersection of the axis with the tangent at the extremity of the ordinate.*



Draw TQ to the circumference perpendicular to AM; and join NQ, CQ.

Then $TY : SY$
And $TZ : HZ$ $\Big\}$ $::$ $TN : PN$

$\therefore$ $TY \times TZ : SY \times HZ\,(BC^2) :: TN^2 : PN^2$

And $BC^2 : AC^2 :: PN^2 : CN^2 - CA^2$

But $TY \times TZ = AT \times TM = QT^2$

Hence $QT^2 : TN^2 :: CA^2 : CN^2 - CA^2$

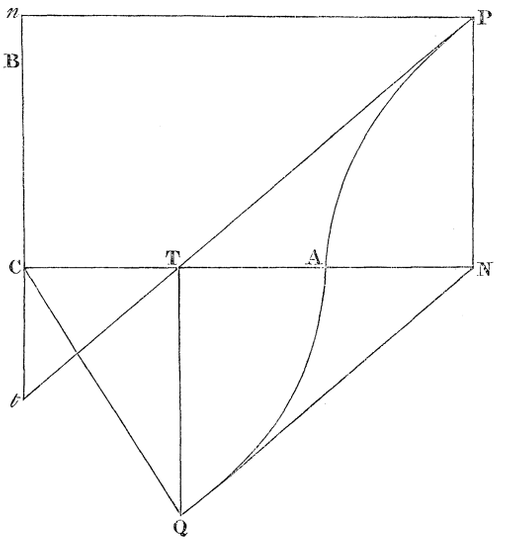
Comp^{do}. $QT^2 : QN^2 :: CA^2\,(CQ^2) : CN^2$.

The triangles QTN, CQN are therefore similar (Eucl. 7. vi.), and $\angle CQN$ is a right angle; and NQ touches at Q. The tangents TP, NQ have therefore the same subtangent TN.

Also $CN : CQ\,(CA) :: CQ\,(CA) : CT$.

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Cor. 1. Draw Pn perpendicular to the axis minor, and produce the tangent PT to meet the axis minor in t.



Then $Ct : PN(Cn) :: CT : NT$

$:: CQ^2 : NQ^2 (CN^2 - CQ^2)$

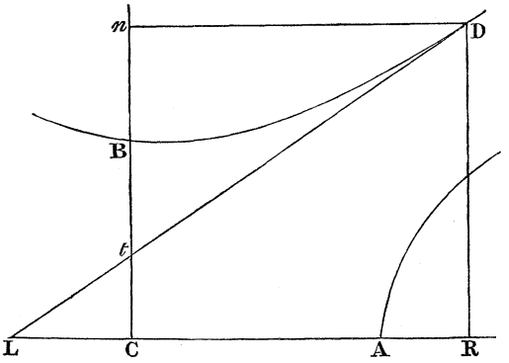
$:: BC^2 : PN^2$ (Prop. 1. Cor. 1.)

Whence $Ct : BC :: BC : Cn$, as for the axis major.

Cor. 2. Let DL, touching the conjugate hyperbola at D, intersect the axes in L, t; draw DR, Dn, perpendicular to the axes.

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Then $Cn \times nt = Cn^2 - Cn \times Ct$ (Euc. 2. II.) $= Cn^2 - BC^2$.



But $CL : Dn\,(CR) :: Ct : nt$

$\therefore\ CL \times CR : Dn^2 :: Cn \times Ct : Cn \times nt$

$:: BC^2 : Cn^2 - BC^2$

$:: AC^2 : Dn^2$ (Prop. 1. Cor. 1.)

Hence $CL \times CR = AC^2$

Cor. 3. $CR \times RL : CL \times CR\,(AC^2)$

$:: RL : CL :: DR\,(Cn) : Ct$

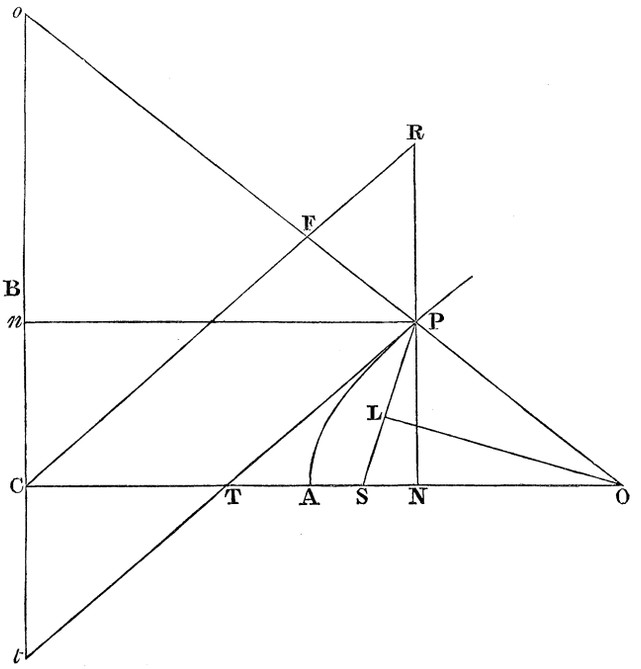
$:: DR^2 : Cn \times Ct\,(BC^2)$.

Hence $CR \times RL : DR^2 :: AC^2 : BC^2$.

Prop. IX.

The rectangle by the normal PO *to either axis and the perpendicular* PF *upon the conjugate diameter is equal to the square of half the other axis.*

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Let PN produced meet the conjugate in R. Then as in the Ellipse, Prop. 9.

$$PO \times PF = PN \times PR = Cn \times Ct = BC^2.$$

Similarly $Po \times PF = CN \times CT = CA^2$.

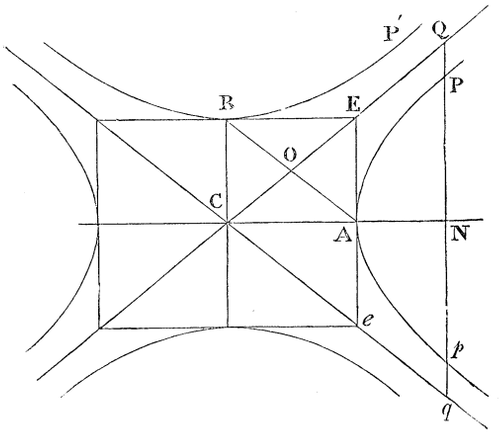
Cor. 1. As in the Ellipse, the subnormal $ON = \frac{L}{2AC} \times CN$ and $on = \frac{2AC}{L} \times Cn$, L being the latus rectum.

Cor. 2. As in the Ellipse, if OL be drawn perpendicular to the focal distance SP, PL = half the latus rectum.

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Prop. X.

If tangents be drawn at the vertices of the axes, the diagonals of the rectangle so formed are asymptotes to the four curves.



Let NP meet the diagonal CE produced in Q.

Then $NQ^2 : CN^2 :: AE^2 (BC^2) : AC^2$

$:: NP^2 : CN^2 - CA^2$ (Pr. 1. Cor. 1.)

Now as CN increases, the ratio of CN^2 to $CN^2 - CA^2$ continually approaches to equality; but $CN^2 - CA^2$ is never actually equal to CN^2, if CN be ever so much increased. Hence NP is always less than NQ, but approaches continually nearer to equality with it.

By the same proof, CQ is an asymptote to the conjugate hyperbola BP'.

Cor. 1. The two asymptotes make equal angles with the axis major, and with the axis minor.

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Cor. 2. The line AB joining the vertices of the conjugate axes is bisected by one asymptote and parallel to the other.

It is bisected by CQ, because the diagonals of the rectangle $BCAE$ bisect each other; and it is parallel to Ce, because $EO = OC$ and $EA = Ae$ (Eucl. 2. vi.)

Cor. 3. All lines perpendicular to either axis and terminated by the asymptotes are bisected by the axis.

Cor. 4. In the rectangular hyperbola, the asymptotes are at right angles to each other.

Prop. XI.

If any line Qq *perpendicular to either axis be terminated by the asymptotes, the rectangle of the segments, into which the curve divides it, is equal to the square of half the conjugate axis.*

For $NQ^2 : CN^2 :: NP^2 : CN^2 - CA^2$ (Prop. 10.)

$\therefore NQ^2 - NP^2 : CA^2 :: NP^2 : CN^2 - CA^2$

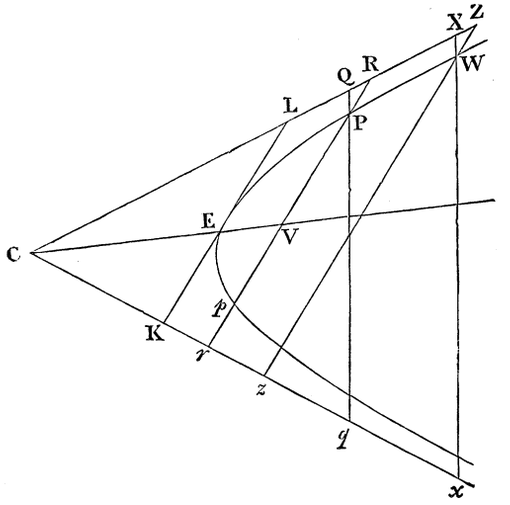
$:: BC^2 : CA^2$ (Prop. 1. Cor. 1.)

Hence $NQ^2 - NP^2$ or $QP \times Pq = BC^2$, and is therefore the same wherever in the curve the point P is taken.

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Prop. XII.

If any line whatever Rr, *making a given angle with either asymptote, cut the curve in* P, *the rectangle by the segments* RP, Pr, *is invariable.*



Through any other point W of the curve draw $Z\varkappa$ parallel to Rr, and through P, W, draw Qq, Xx perpendicular to the axis. Then, by similar triangles,

$RP : QP :: ZW : XW$

$Pr : Pq :: W\varkappa : Wx$,

$\therefore RP \times Pr : QP \times Pq :: ZW \times W\varkappa : XW \times Wx$.

But $QP \times Pq = XW \times Wx$ (Prop. 11.)

$\therefore RP \times Pr = ZW \times W\varkappa$.

Cor. 1. Let Rr move parallel to itself until the points P, p, coincide, as at E. Then LEK touches at E, and $RP \times Pr = LE \times EK$. Also by the same proof $Rp \times pr = LE \times EK$.

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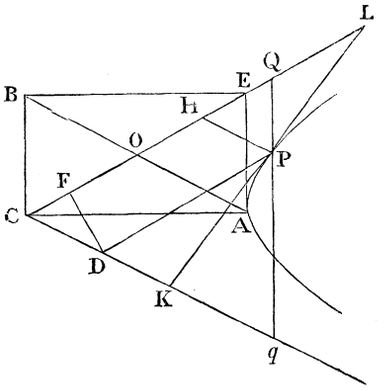
Cor. 2. $RP \times Pr = Rp \times pr$, or $RP \times Pp + RP \times pr = RP \times pr + Pp \times pr$ (Eucl. 1. II.): whence $RP = pr$.

Cor. 3. $LE = EK$, and $RP \times Pr = LE^2$.

Cor. 4. Let the diameter CE cut Rr in V: then $VR = Vr$, by similar triangles, and $RP = pr$; whence $PV = Vp$; *i.e.* a diameter bisects all it's own ordinates.

Prop. XIII.

If from any point P *in the curve straight lines* PD, PH, *be drawn parallel to the asymptotes, their rectangle is invariable.*



Draw the tangent LPK, and Qq perpendicular to the axis. Then, by similar triangles,

$PH : PQ :: AO : AE$

$PD : Pq :: OE : AE$

$\therefore PD \times PH : PQ \times Pq :: AO^2 : AE^2$

But $PQ \times Pq = AE^2$ (Prop. 11.) $\therefore PD \times PH = AO^2 = \frac{1}{4}(AC^2 + BC^2)$ a constant quantity.

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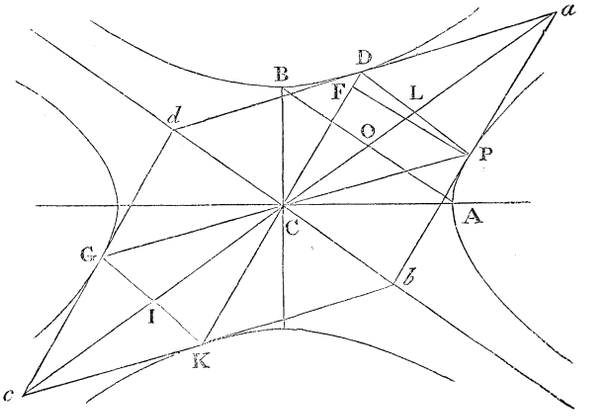
Cor. 1. Draw DF perpendicular to CH; then the triangle DCF is always similar to itself. Hence $CD \times CH$ and $DF \times CH$ are in a constant ratio; and therefore $DF \times CH$ or the parallelogram DH is a constant quantity.

Cor. 2. The triangle CLK is a constant quantity, being double of the parallelogram DH.

Cor. 3. $PH \propto \frac{1}{CH}$.

Prop. XIV.

If a straight line PLD *be drawn parallel to one asymptote, and terminated by the conjugate hyperbolas, it is bisected by the other asymptote.*



For $CL \times LP = AO^2$ (Prop. 13.) $= BO^2 = CL \times LD$;
$\therefore PL = LD$.

Cor. 1. If ab touch at P, it is bisected in P (Prop. 12. Cor. 3.); therefore $Ca = 2CL$ (Eucl. 2. vi). For this reason a tangent dD at D must meet Ca

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in the same point a. Also since $PL = LD$, $Cb = Cd$, by similar triangles; and Cb or $Cd = 2LP = PD$. Hence CD is parallel to ab, and CP to ad (Eucl. 33. 1.)

Cor. 2. Draw bKc touching at K, and from c draw cGd touching at G. Then cb, cd being bisected at K, G, GK is parallel to bd and bisected in I. Hence, as in the last Cor^{y}, $Cd = Cb$, and ad, cd, meet Cd in the same point d. Also $IK = \frac{1}{2}Cb = LD$, whence $CI = CL$. (Prop. 13. Cor. 3.) The triangles CIK, CLD are therefore equal in all respects, and in like manner the triangles CLP, CIG. Hence PCG, DCK are straight lines and conjugate diameters; they are likewise bisected by the center C.

Cor. 3. The tangent ad is equal and parallel to the diameter PG, and ab to the conjugate DK.

Cor. 4. The figure $abcd$ is a parallelogram, of which $CPaD$ is a fourth part.

Cor. 5. In the rectangular hyperbola, every diameter PCG is equal to it's conjugate DCK.

Prop. XV.

The parallelograms formed by tangents at the vertices of any pair of conjugate diameters have all the same area.

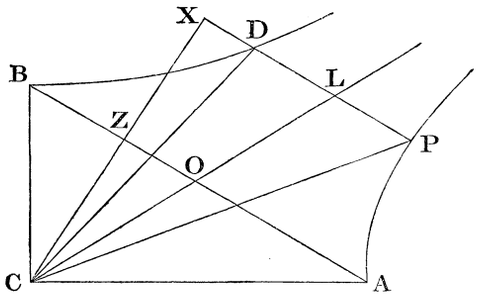
For the parallelogram $CPaD = 2\triangle CPa = \triangle Cab$ (Eucl. 38. 1.) a constant quantity (Prop. 13. Cor. 2.)

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Cor. Draw PF perpendicular to CD; then the parallelogram $CPaD = CD \times PF$. Now when the tangents are drawn at A and B, CD coincides with CB, and PF with AC. Hence $CD \times PF = AC \times BC$.

Prop. XVI.

The difference of the squares of any two conjugate diameters is equal to the difference of the squares of the axes.



Draw CZX perpendicular to AB and PD,

Then $CP^2 - CD^2 = PX^2 - DX^2$

$= 4PL \times LX$ (Eucl. 8. II.)

But $LX : OZ :: CL : CO$, by sim. $\triangle^s$.

$:: AO : PL$ (Prop. 13. Cor. 3.)

$\therefore 4PL \times LX (CP^2 - CD^2) = 4AO \times OZ$

$= AC^2 - BC^2$.

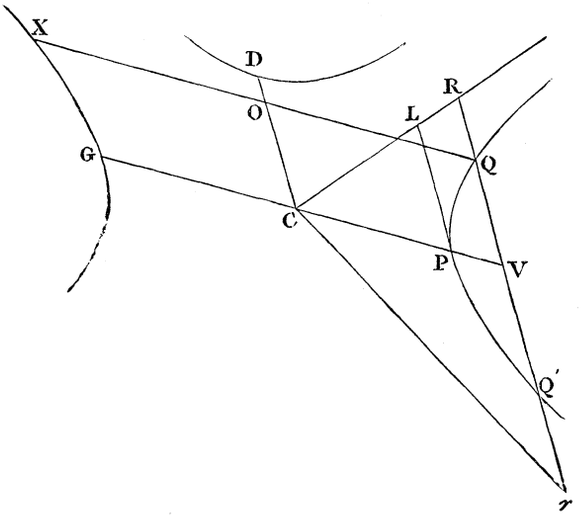
Cor. As in the Ellipse, Prop. 11. Cor. 2, the rectangle by the focal distances is equal to the square of the semi-conjugate diameter.

Prop. XVII.

The rectangle by the abscissæ PV, VG, *of any diameter* PG *is to the square of their semi-ordinate* QV *as the square of* CP *to the square of the semi-conjugate* CD.

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Let QQ' meet the asymptotes in R, r. Then



$RQ \times Qr$ or $RV^2 - QV^2 = PL^2$ (Prop. 12. Cor. 3.)
$= CD^2$ (Prop. 14. Cor. 3.) $\therefore QV^2 = RV^2 - PL^2$;
But $CV^2 : CP^2 :: RV^2 : PL^2$
Div^{do}. $CV^2 - CP^2 : RV^2 - PL^2 :: CP^2 : PL^2$
Or $PV \times VG : QV^2 :: CP^2 : CD^2$.

Cor. 1. $QV^2 \propto PV \times VG \propto CV^2 - CP^2$.

Cor. 2. Let QOX, parallel to PG, cut CD in O and the opposite hyperbola in X.

Then $CV^2 - CP^2 : CP^2 :: QV^2 : CD^2$
Comp^{do}. $CV^2 : CD^2 + QV^2$
Or $QO^2 : CD^2 + CO^2$ $\Big\}$ $:: CP^2 : CD^2$
And so $XO^2 : CD^2 + CO^2$

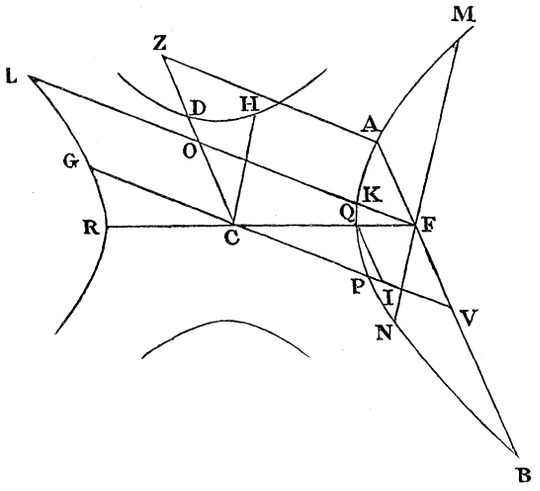
Hence $QO = OX$; *i. e.* every line terminated by two opposite hyperbolas is bisected by that diameter to whose conjugate it is parallel.

Cor. 3. $CV^2 \propto CD^2 + QV^2$; and $QO^2 \propto CD^2 + CO^2$.

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Prop. XVIII.

If any line AB *intersect a diameter* RQ *produced in* F, *and* CD *be the semi-diameter parallel to* AB, *then* $AF \times FB : QF \times FR :: CD^2 : CQ^2$.



Let CPV be the diameter bisecting AB, and draw QI parallel to AB; then (Prop. 17. Cor. 3.)

$$CD^2+AV^2 : CD^2+QI^2 :: CV^2 : CI^2$$

Div^{do}. $AV^2-QI^2 : CD^2+QI^2 :: CV^2-CI^2 : CI^2$

$$:: FV^2-QI^2 : QI^2 \text{ by sim. } \triangle^s.$$

Hence $AV^2-FV^2 : CD^2 :: FV^2-QI^2 : QI^2$ (E. 19. v.)

$$:: CF^2-CQ^2 : CQ^2 \text{ by sim. } \triangle^s.$$

Or $AF \times FB : CD^2 :: QF \times FR : CQ^2$.

Cor. 1. If any other line MN cut AB in F, and CH be the semi-diameter parallel to MN, then $AF \times FB : MF \times FN :: CD^2 : CH^2$.

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Cor. 2. Draw FKL parallel to VG, and cutting the opposite hyperbolas; also draw AZ parallel to VG: then (Prop. 17. Cor^{s}.)

$AZ^2\ (FO^2) : KO^2 :: CD^2 + CZ^2 : CD^2 + CO^2$

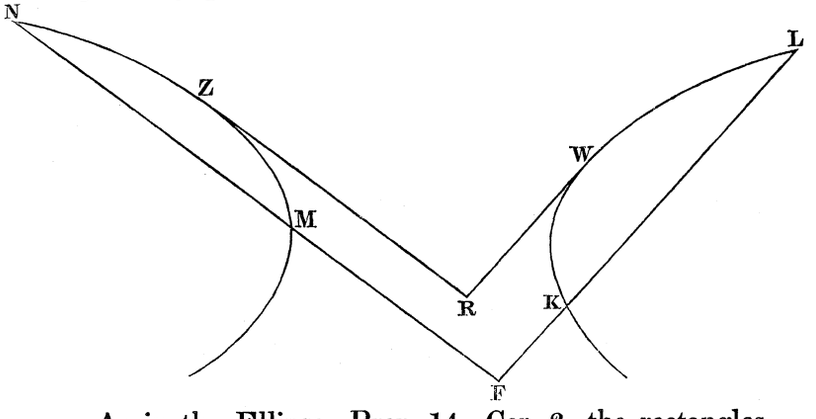
Div^{do}. $FO^2 - KO^2 : CZ^2 - CO^2 :: KO^2 : CD^2 + CO^2$

Or $KF \times FL : AF \times FB :: CP^2 : CD^2$

$\therefore\ KF \times FL : MF \times FN :: CP^2 : CH^2$.

Cor. 3. If AB, KL, MN, move parallel to themselves, and intersect in F either within or without the hyperbola, the rectangles $AF \times FB$, $KF \times FL$, $MF \times FN$, have always the same ratio to each other as the squares of the diameters to which the lines are respectively parallel.

Cor. 4. If from any point R tangents RW, RZ be drawn to either of the opposite hyperbolas, these are to each other as the diameters to which they are respectively parallel.



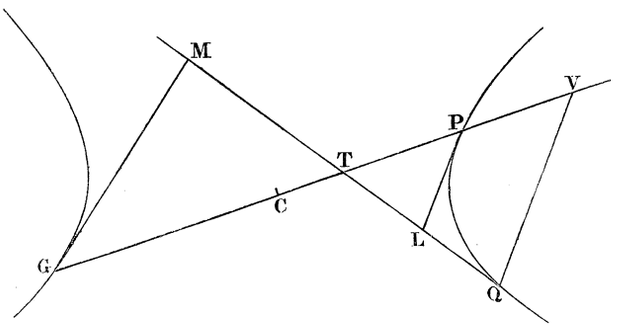
As in the Ellipse, Prop. 14. Cor. 3, the rectangles $FK \times FL$ and $FM \times FN$, and therefore the squares of RW and RZ, are as the squares of the parallel diameters.

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Prop. XIX.

If a tangent QT *at the extremity* Q *of any ordinate* QV *meet the diameter* CP *to which the ordinate belongs in* T; *then* CP *is a mean proportional between* CV *and* CT.

At the extremities P, G of the diameter let the tangents PL, GM meet the tangent QT in L, M.



Then $MQ : MG :: LQ : LP$ (Prop. 18. Cor. 4.)
And therefore $MQ : LQ :: MG : LP$
Hence $VG : PV :: TG : TP$ by sim. Δ^s.
$\therefore VG+PV : VG-PV :: TG+TP : TG-TP$
Or $2CV : 2CP :: 2CP : 2CT$.

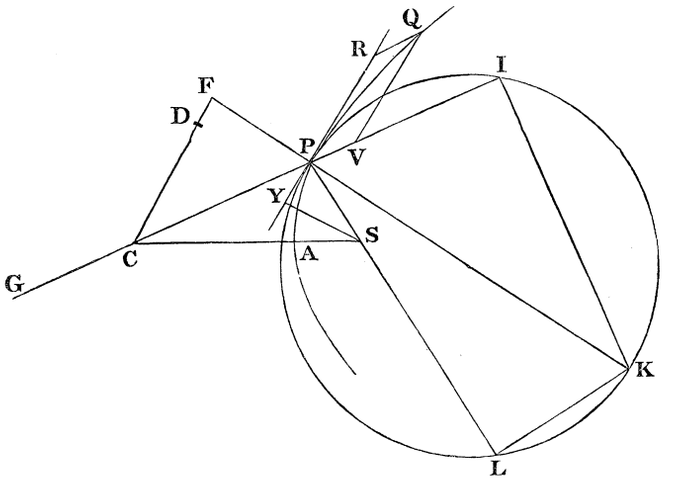
Cor. 1. A tangent at the other extremity of the ordinate would meet the diameter CP in the same point T.

Cor. 2. $PV \times VG = CV^2 - CP^2 = CV^2 - CV \times CT = CV \times VT$.

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Prop. XX.

To determine the diameter of curvature and the chords through the center and focus of an hyperbola.



As in the Ellipse, Prop. 17.

I. The chord PI through the center $= \frac{2CD^2}{CP}$.

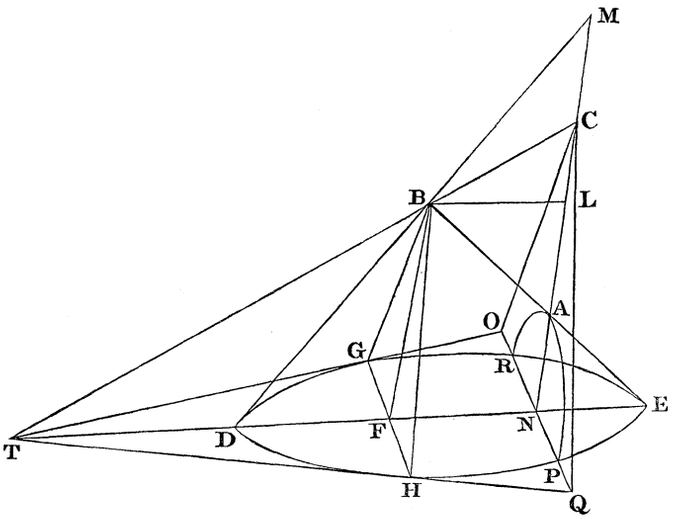
II. The diameter $PK = \frac{2CD^2}{PF}$, or $= L \times \frac{SP^3}{SY^3}$.

III. The chord PL through either focus $= \frac{2CD^2}{AC}$.

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Prop. XXI.

If BDE *be a cone, from which an hyperbola* RAP *is cut;* DGEH *a circular section perpendicular to the axis of the cone;* BGH *a triangular section through the vertex* B *parallel to* RAP; BTG, BTH *planes touching the cone along the slant sides* BG, BH; *then* CO, CQ, *the common sections of* BTG, BTH, *with the cutting plane* RAP, *are the asymptotes of the hyperbola.*



Draw ***BL*** parallel to ***DE***, meeting ***AM*** in ***L***. Then the axes of the hyperbola being in the proportion of ***BF*** to ***FH***, the ∠ ***GBH*** or the equal ∠ ***OCQ*** (Eucl. 16. xi.) is the angle between the asymptotes.

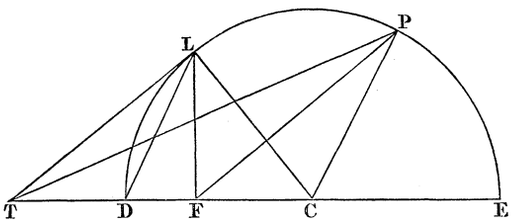
Now by similar triangles ***ALB***, ***BFE*** and ***CLB***, ***BFT***, ***AL*** : ***CL*** :: ***TF*** : ***FE*** and therefore ***AC*** : ***CL*** :: ***TE*** : ***FE***. In like manner by similar triangles

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MLB, BFD, and *CLB, BFT*, *ML* : *CL* :: *TF* : *DF* and therefore *CM* : *CL* :: *TD* : *DF*. But by the property of the circle*, *TE* : *FE* :: *TD* : *DF*. Therefore *CA* = *CM* (Eucl. 9. v.). Hence *C* is the center of the hyperbola, and *CO*, *CQ* are the asymptotes.

---

* If *DLE* be a circle; *T* any point in the diameter produced; *TL* a tangent at *L*; *LF* perpendicular to the diameter; *P* any other point in the circumference; then *TP* : *PF* is in the constant ratio of *TL* : *LF*.



Join *CL*, *CP*. Then *CT* : *CL* (*CP*) :: *CL* (*CP*) : *CF* (Eucl. 8. VI.), and the Δ^s *TCP*, *FCP*, are similar, (Eucl. 5. VI.) Hence

TP : *PF* :: *CP* (*CL*) : *CF* :: *TL* : *LF*.

Therefore *TD* : *DF* and *TE* : *EF* are in the same ratio.

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ON

THE SECTIONS

OF

The Conoids.

DEFINITIONS.

1. THE solids generated by the revolution of conic sections about their axes are called *Conoids.*

2. If a Parabola so revolve, the solid is called a *Paraboloid.*

3. If an Ellipse revolve about it's axis major, the solid is called a *Prolate Spheroid;* and if it revolve about the axis minor, an *Oblate Spheroid.*

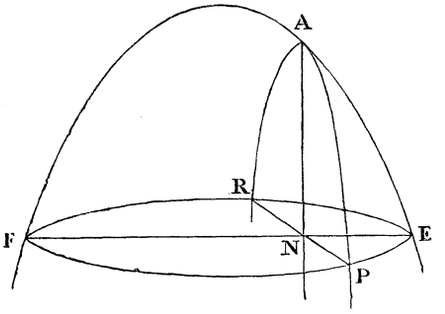
4. The solid generated by the revolution of an Hyperbola about it's axis major is called an *Hyperboloid.*

SCHOLIUM. Every ordinate to the axis of revolution describes a circle whose center is in the axis; therefore all sections, whose planes are perpendicular to the axis, are circles. Also the revolving plane, since it always passes through the axis, is perpendicular in every position to the planes of all the circular sections.

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Prop. I.

If a paraboloid be cut by a plane parallel to the generating plane, the section is the same as the parabola which revolves.



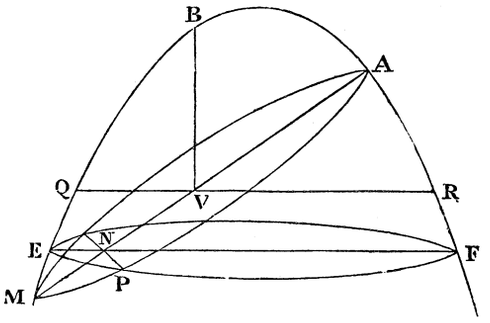
Let FAE be that position of the generating parabola which is perpendicular to the cutting plane RAP, and $FPER$ any circular section. Then both the planes FPE, RAP are perpendicular to FAE, and therefore PNR is perpendicular to FAE and to the lines AN, EF. Also AN is parallel to the axis of the solid.

If L be the latus rectum of the revolving parabola, $L \times AN = EN \times NF$ (Parabola, Prop. 9.) $= NP^2$. Wherefore the curve RAP is the parabola whose latus rectum is L.

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Prop. II.

Any other section of a paraboloid is an ellipse.



Let the generating plane MBF be perpendicular to the cutting plane AMP, AM their common section, which cutting the parabola in one point A, must cut it again, as at M (Parabola, Prop. 3. Cor. 1.). Also let BV be the diameter to which AM belongs, and draw QVR an ordinate of the axis.

Then (Parabola, Prop. 12. Cor. 2.)

$AN \times NM : EN \times NF :: AV \times VM : QV \times VR$

Or $AN \times MN : NP^2 :: AV^2 : L \times BV$.

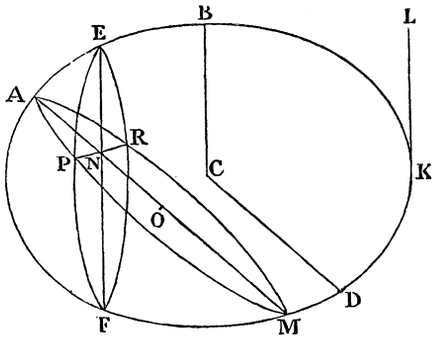
The curve is therefore an ellipse, whose major axis is AM, and whose semi-axis-minor is a mean proportional between L and BV.

Prop. III.

All sections of a spheroid, except those which are perpendicular to the axis, are ellipses.

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Let ABD represent a *prolate* spheroid, CB the



semi-axis minor of the revolving ellipse, CD a semi-diameter parallel to AM: then (Ellipse, Pr. 14. Cor. 1.)

$$AN \times NM : EN \times NF\,(NP^2) :: CD^2 : CB^2.$$

The curve is therefore an ellipse, whose axis major is AM, and whose axis minor is to AM as CB to CD.

If ABD represent an *oblate* spheroid, the circular section EPF must be made perpendicular to BC; and by the same demonstration APM is an ellipse, whose axis *minor* is AM.

Cor. 1. All sections whose planes are parallel to each other, are similar ellipses, because CB and CD remain unaltered.

Cor. 2. If ABD be an *oblate* spheroid, and KL touch the generating ellipse at the extremity K of the axis major, all sections made by planes passing through KL are similar to the generating ellipse. For KL is parallel to CB and therefore the plane of every such section is parallel to the generating plane in some one position of it.

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